Influence of Mass on the Damping of a Pendulum

Pablo Morandé (s1976852)
Lab Partner: Miguel Pérez Merelles

March 2021

Contents

Abstract

The effect of mass on the damping of a pendulum was verified by analysing several videos of pendulums with different masses. The observed data for each ball was compared to the provided model to refine the parameters of the motion for each case and to obtain a best estimate of γ, which causes the damping. The different values of γ were used to obtain a best estimate of b, the damping parameter which depends only on the geometrical properties of the Balls (which were identical in that sense). The experimentally determined value of b was $1.620(12) \cdot^{-4} \frac{kg}{s}$. However, this value was found to disagree with the theoretical expectations by one order of magnitude. Such discrepancy might arise for choosing an unsuitable model of friction to describe the data.

1 Introduction and Theory

The simple pendulum is usually used to introduce students to Simple Harmonic Oscillators, also the damped pendulum is used to introduce University level students into the topic of damped harmonic motion.
The aim of this experiment was to explore the given model for the damped pendulum and to determine the effect of the mass of the pendulum on the damping. The supplied model gave the following expression for the horizontal displacement:

$$x = A_0 e^{(-\gamma t)} \cos(\omega t + \phi) \tag{1}$$

In this equation A_0 is the initial displacement, $\gamma = \frac{b}{2m}$ where b is the damping parameter and m is the mass of the ball, ω is the frequency of the motion and it relates with the natural frequency by $w = \sqrt{w_0^2 - \gamma^2}$ and ϕ is an offset that it is added to account for the initial position as $x_{t=0} = A_0 cos(\phi)$ There are three forces acting on the pendulum: gravity, tension and air resistance.

1.1 Mathematical Derivation

To understand the derivation of the damped pendulum it is useful to first understand the equation of the simple pendulum.
In this case there are two forces acting on the pendulum, the tension and the weight of the ball. It is easier to express this problem in polar coordinates. Using a polar basis the two unit vectors transform into the radial vector $\underline{e_r}$ and the tangent vector $\underline{e_\theta}$, this two vectors relate to the normal Cartesian vectors[1] by;

$$\begin{aligned}
\underline{e_r} &= \cos(\theta)\underline{e_1} + \sin(\theta)\underline{e_2} \\
\underline{e_\theta} &= -\sin(\theta)\underline{e_1} + \cos(\theta)\underline{e_2} \\
x_1 &= r\cos(\theta) \\
x_2 &= r\sin(\theta)
\end{aligned}$$

Using Polar coordinates the position vector, the velocity vector and the acceleration vector are transformed into.

$$\begin{aligned}
\underline{r} &= r\underline{e_r} \\
\underline{\dot{r}} &= \dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta} \\
\underline{\ddot{r}} &= (\ddot{r} - r\dot{\theta}^2)\underline{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e_\theta}
\end{aligned}$$

The only force acting on the tangent direction is part of the weight, the force acting in the direction of the tangent of the motion is[1]:

$$F_\theta = -mg\sin(\theta)$$

So applying Newton's Second Law in the tangent direction we get

$$-mg\sin(\theta) = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) - g\sin(\theta) = r\ddot{\theta}$$

[1] In this case $\underline{e_1}$ points vertically down and $\underline{e_2}$ points in the horizontal direction (right), so x_2 is the horizontal direction in this case, $x_2 = x$

This step is possible as the radius is constant and equal to the length of the pendulum. Using the small angle approximation $\sin(\theta) = \theta$ so and as $\sin(\theta) = \frac{x}{L}$, in this case x is defined as the horizontal displacement:

$$\begin{aligned} \theta &= \frac{x}{L} \\ -g\theta &= L\ddot{\theta} \\ \frac{-g}{L}x &= \ddot{x} \\ w_0^2 &= \frac{g}{L} \\ -w_0^2 x &= \ddot{x} \end{aligned}$$

Obtaining the equation that describes the damped pendulum is now fairly easy, applying the same techniques used to derive the equation that governs the simple pendulum. The only difference is that there is a new force in the tangential direction (direction of motion) that is the air resistance, assuming a linear model the force on the tangential direction is given by:

$$F_\theta = -mg\sin(\theta) - b\dot{x}$$

Taking Newton's second Law in the tangential direction an applying $\sin(\theta) = \theta$ so and as $\sin(\theta) = \frac{x}{L}$:

$$\begin{aligned} m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= -mg\sin(\theta) - b\dot{x} \\ mL\ddot{\theta} &= -mg\sin(\theta) - b\dot{x} \\ m\ddot{x} &= \frac{-mg}{L}x - b\dot{x} \end{aligned}$$

This leads to the equation:

$$m\ddot{x} + \frac{mg}{L}x + b\dot{x} = 0$$

Which governs the motion of the damped pendulum (damped harmonic oscillator[1]). To solve this equation it is useful to introduce new variables

$$w_0^2 = \frac{g}{L}\gamma = \frac{b}{2m}$$

Using this new variables the equation becomes:

$$\ddot{x} + \omega_0^2 x + 2\gamma\dot{x} = 0$$

The method to solve this linear differential equation is to consider a trial solution of the form $x = e^{\lambda t}$[1]. Calculating $\ddot{x}$ and $\dot{x}$ with the trial solution and substituting back into the original equation yields:

$$\begin{aligned} \lambda^2 e^{\lambda t} + \omega_0^2 e^{\lambda t} + 2\gamma\lambda e^{\lambda t} &= 0 \\ \lambda^2 + w_0^2 + 2\gamma\lambda &= 0 \end{aligned}$$

This equation can be solved for λ, giving:

$$\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} = -\gamma \pm \sqrt{(\omega_0^2 - \gamma^2)}i = -\gamma \pm \omega i$$

Since there are two linearly independent solutions for λ, and as the equation is a linear differential equation, a linear combination of both solutions is also a solution. Therefore the general solution for the damped pendulum is:

$$x(t) = e^{-\gamma t}(Ae^{\omega i t} + Be^{-\omega i t}) = A_0 e^{-\gamma t}\cos(\omega t + \phi)$$

-	Ball 1	Ball 2	Ball 3	Ball 4	Ball 5
mass(g)	1.86(8)	5.71(8)	15.63(8)	34.51(10)	65.60(8)

Table 1: Shows the masses of the Balls constructed using the given materials for this experiment

1.2 Linear Drag

The proposed model assumes that the Drag Force is proportional to the velocity such that $F_D = -bv$, this model will hold if the Reynolds number of the ball of the pendulum is less than one [2] when it is moving through fluid. If this holds then the value of b can be estimated to be $b = 6\pi r\eta$ [3].

2 Experimental Methods

2.1 Materials

This sub section presents the materials used in the experiment (including software)

- Scale, it allowed to measure from 12 cm to -12cm
- 4 Steel Balls of radius:
  10mm, 15mm, 20mm, 25mm
- 2 Polystyrene Balls split into half and with a hole inside to fit the Steel Balls, the radius of the Balls was 50mm
- Phone, was used to record the videos
- Computer, was used for the analysis of the videos and the data
- VideoPad, software used to analyse the videos
- thread, used to construct the pendulum of 1.5[m]
- Rubber bands, used to hold together the two ends of the Polystyrene Ball

The Steel Balls are combined with the Polystyrene Balls and the rubber bands to create complete balls with different masses that which were used in the experiment. For simplicity those "complete" balls are denoted as Ball 1,2,3,4,5. In each case a different steel ball was fit into the polystyrene ball and it was held together by 2 rubber bands in the case of the first 3 and by only one in the last two. These balls are ordered by masses in decreasing order and the fifth ball contained no steel ball, it was just constructed by the polystyrene ball with the smaller hole. For the first 3 balls the polystyrene with the bigger hole was used and for the last two the polystyrene ball with the smaller hole was used. The final masses of the balls accounting for all the components were determined to be[2]:

2.2 Experimental Procedure

2.2.1 Set-Up

The provided thread was used in combination with the Polystyrene Balls to construct a pendulum of an approximate longitude of 1.5 metres. The rubber bands were used to hold together the two ends of the polystyrene balls (which had the steel balls inside, one different in each video). The scale was placed so that it was feasible to read the position of the end of the thread, also the position 0 of the scale was placed in the equilibrium position of the pendulum.
All the data were collected using the same set up, and the variations (of the set up) between balls were minimum. To start the oscillations the ball was placed in the +12cm position and it was released without

[2]The masses of the balls were found by adding the masses of the individual components that form the ball (steel ball, polystyrene ball and rubber bands), the error on the mass of the ball each ball was determined by propagation of errors [4]. The error on each individual component was taken to be the standard deviation, calculated with the provided data set

adding any force into the motion.

2.2.2 Data Collection

After making the set-up, one video was recorded for each of the balls. The duration of the videos varied from almost 20 minutes for the heaviest ball to about 4 minutes for the video without any steel Ball. Therefore, slightly different methods were used to extract the data, as the same number of data points were needed to perform the analysis on each ball.
After recording all the videos they were exported to VideoPad. This software allows to analyse the data frame by frame and it also gives the time at which the frame was recorded up to the nearest millisecond. This experiment required two types of data: first, the data of all the balls in the first 2-3 periods and second, the data of all the balls for long periods of time. The first type of data for each ball was used to create an approximate model of the pendulum without damping, as the damping is not noticeable for most balls in the first three periods. The second part of the data, allowed to refine the model including damping, as the decay of the amplitude was only observed over long periods of time.
To collect the data of the first 2-3 periods for all the balls the movies were analysed frame by frame and the position of the ball was recorded each 0.1 seconds. To extract the real time of the motion, it was required to subtract the time at which the ball starts to oscillate to each data point (this defines the zero in the movement).
Nevertheless, this approach is not feasible to extract the data for the long periods of time for all the balls, as the duration of the videos is too long and analysing the full video using this technique would give about 12000 data points for the heaviest ball. Therefore, it was decided that the best way of measuring the decay was to measure the maximum amplitude (positive and negative) each n periods of oscillation. The number n varied from ball to ball, and it was chosen so that at the end the total number of data points for each ball was around 40 for the long period data.
Some Difficulties were found when collecting the data:

1. In the first set of videos, the initial positions of the balls were not clear, as the balls were in a position where it hindered the measurement. The videos were retaken with the balls in a more appropriate position relative to the scale to facilitate the Data collection.

2. For the third ball it was found that the amplitude after the first swing was bigger than the initial position, this might have been caused by the ball being pushed a small amount when released, adding energy to the system. To solve this issue the data for this ball was taken from the third period onward.

3. The videos were first analysed only up to the point where the amplitude of the motion was half of the initial amplitude. However, this data was found insufficient to have a proper decay patter in the data, specially for the lighter balls. This was addressed by retaking the measurements and extending it to the point were the amplitude of the motion was only about 1 cm (0.5cm in some cases).

Some Sources of Error that should be considered:

1. Some of the balls ended the motion performing a circular motion in the z-x plane, this could have distorted the data at the end of the motion for such videos

2. It was not possible to set the initial amplitude to 12 cm accurately.

3. The position of the ball is taken to be the the position of the end of the string on the scale. This was done as it was not possible to determine accurately the position of the centre of mass on the scale. However, it might affect the quality of the data.

4. The position of the Ball could only be measured up to 0.5 cm accurately. This means that all the values recorded were either integers or an integer plus 0.5. This could have cause some error in the data due to the inability of distinguishing two different data points. This was more noticeable in the case of the lighter balls, which decayed faster.

3 Results and Analysis

3.1 Short Data

After collecting the data for the five balls it was analysed to obtain a value of γ in each motion (as γ is mass dependent).
To obtain the best estimate of γ in each case a two step optimisation procedure was performed. For each ball first it was analysed the short data (2-3 first periods) and it was compared with the positions that would be obtained using the provided model with gamma set to zero, (this was done as it is expected that the damping is not noticeable on the first oscillations of each ball). At this stage solver[3] was used to minimise ChiSq by varying the initial amplitude, the longitude of the pendulum and the phase. Minimising ChiSq is useful as it gives the best fit of the model for the data by varying the initial conditions.
The parameters obtained in this optimisation were used as starting points for the analysis of the combined data in all the balls. This two step procedure is used to avoid getting stuck in a local minimum when using solver.
As an example the data for the 2-3 first oscillations of the heaviest ball and the best fit found by solver is plotted below.

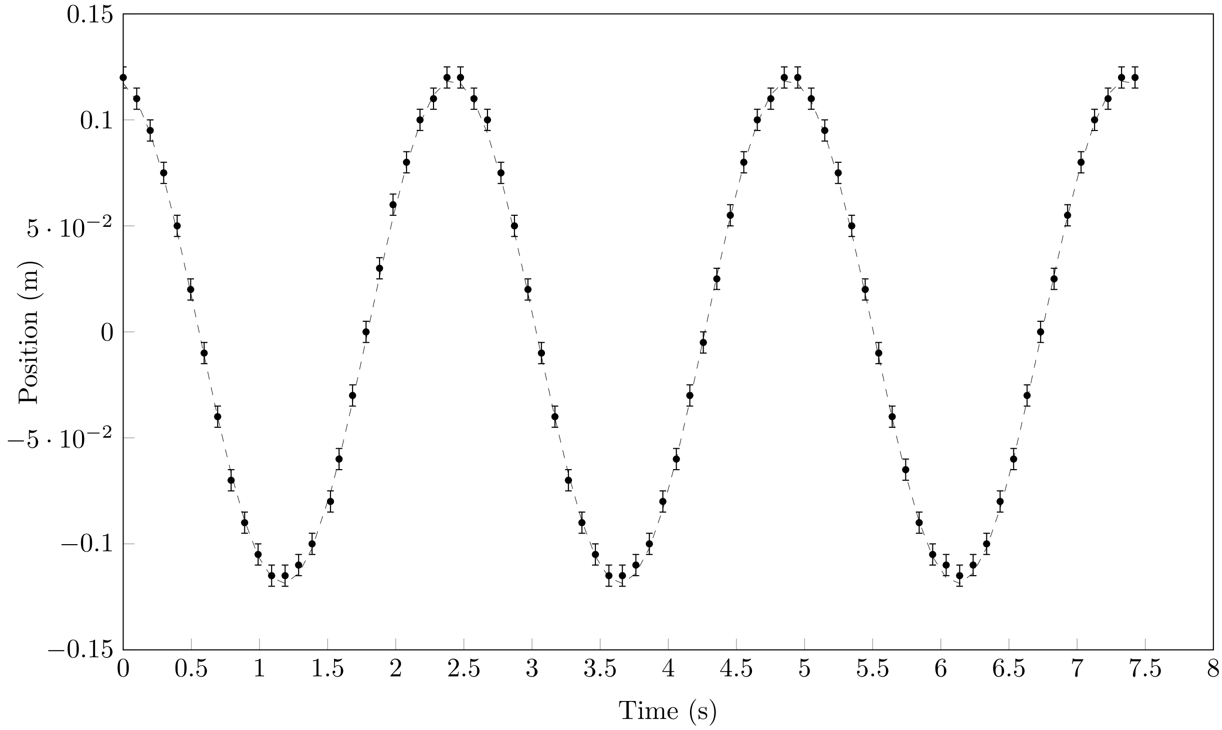

Fig. 1: Graph of the position of the pendulum each 0.1 seconds for the first 3 periods of Ball 1. The uncertainties in the position is 0.005m and the uncertainty in time 0.001 seconds, the latter ones are smaller than the symbols used to plot the data. The dashed line represents the best fit of the model after using solver to minimise ChiSq

It is important to note, that the model with gamma set to zero gave a good fit for all the balls but the lightest one. For this case, the damping was noticeable in the first periods of oscillation so such discrepancy was expected. The discrepancy between the data and the model for the lightest model is shown in figure 2

[3]Solver is an excel package which allows to perform a weighted non-linear fit by varying the selected parameters

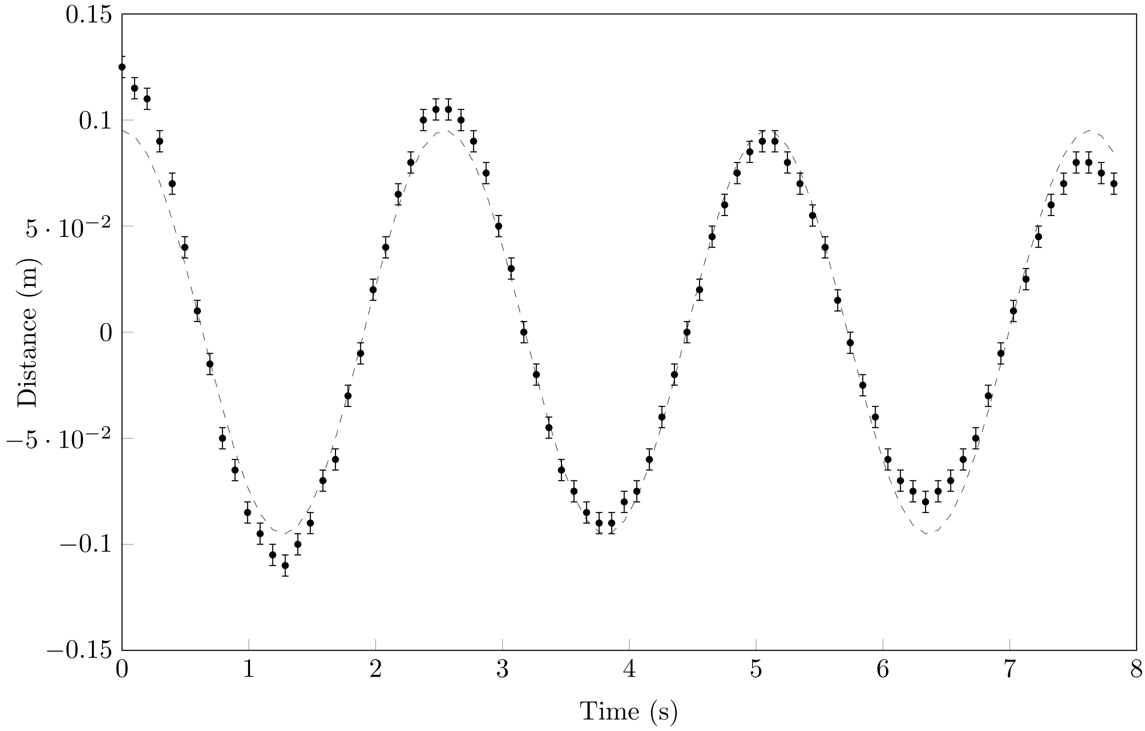

Fig. 2: Graph of the position of the pendulum each 0.1 seconds for the first 3 periods of Ball 5. The uncertainties in the position is 0.005m and the uncertainty in time 0.001 seconds, the latter ones are smaller than the symbols used to plot the data. The dashed line represents the best fit of the model with $\gamma = 0$ after using solver to minimise ChiSq, it can be seen that the model does not fit the data

This analysis was performed for each ball, obtaining 5 sets of good initial parameters to analyse the long period data of the five balls. The initial parameters for each ball where:

-	Ball 1	Ball 2	Ball 3	Ball 4	Ball 5
$A_0(cm)$	11.87	12.04	11.37	10.81	9.5
$L(m)$	1.52	1.54	1.54	1.53	1.60
$\phi(rad)$	0.16	0.01	-0.05	0.03	-0.03

Table 2: Shows the results obtained for the parameters after optimising the model for the short Data

3.2 Combined Data Analysis

After determining the initial parameters the data for the long periods of time was combined with the short data to create a single data set for each ball.
At this stage another parameter was added to the model described in Equation 1, the offset or δ (which would be refined by solver and was initially set to 0). The final model was therefore:

$$x = A_0 e^{(-\gamma t)} \cos(\omega t + \phi) + \delta$$

This new free parameter was introduce as it was seen to improve the agreement between the observed data and the model by reducing ChiSq about 10 units (after its value being refined by Solver).
Solver was used to minimise ChiSq in each data set by varying all the parameters of the model (γ,ϕ,A_0 and L). Therefore, the key parameters of the motion were refined using solver and more importantly γ was

determined for each motion. A comparison between the obtained data and the points calculated by the model is shown below for the first of the five balls.

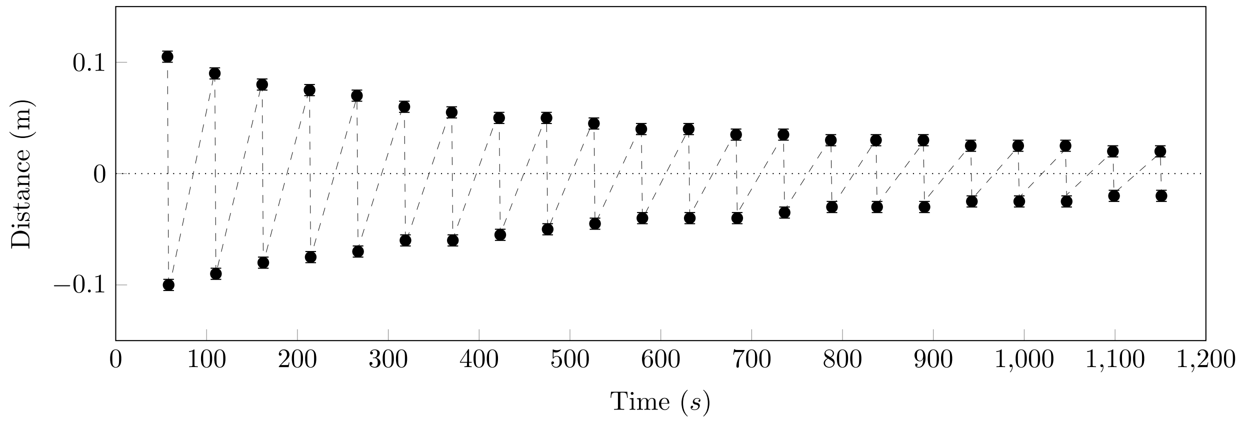

Fig. 3: Graph of distance vs time which displays the data over a long period obtained for the first ball. The dashed line is the result of joining the points calculated by the model. The uncertainties on each data point are ± 0.005m in the vertical axis and the uncertainties on the time are smaller than the symbols used to plot the data. It is clear that the maximum amplitude of the motion decays over time

Fig. ?? represents an example of how the obtained data for all the balls compares to the calculated by the model, this plot was typical for all the balls so it was decided to include just one instance of the five of them. From the graph it was clear that the amplitude of the motion was not longer a constant (as in **Fig. ??**) and that it decayed over time. Therefore, it was important to verify how the maximum amplitude of the motion evolves according to the model after refining the parameters for each ball compared to the obtained data for each ball. The graphs containing that information are shown below.

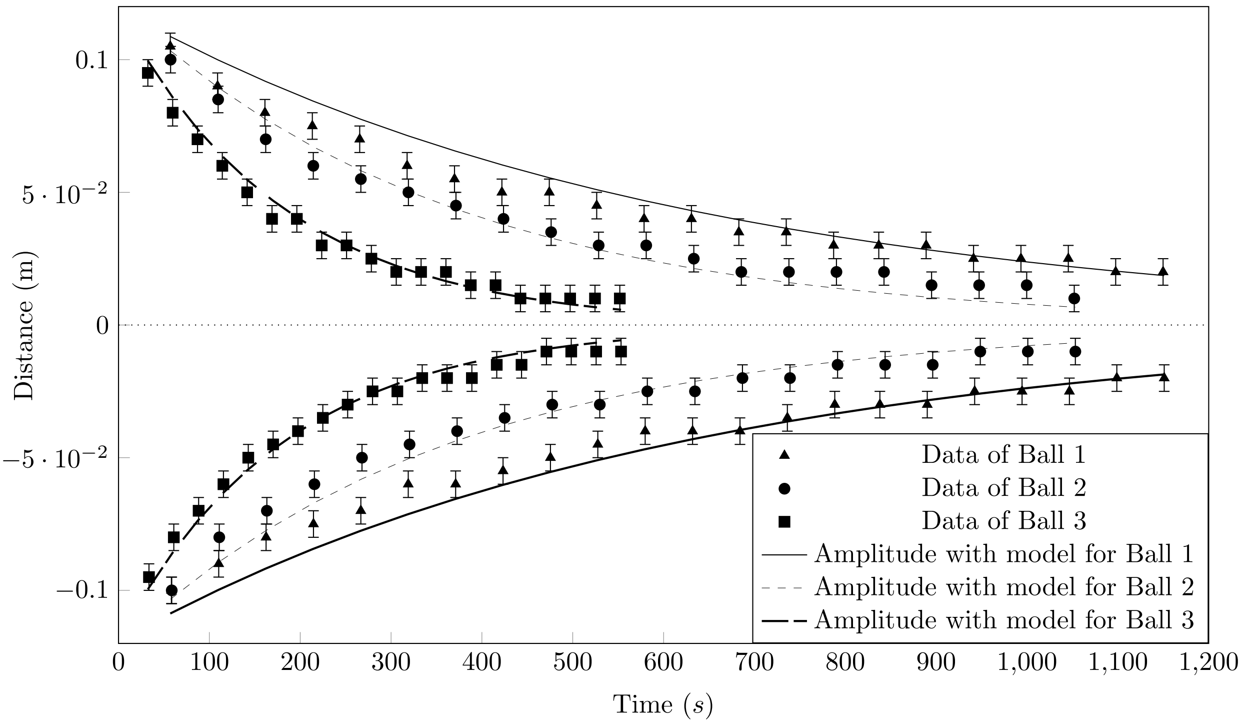

Fig. 4: Graph of distance vs time of the first three balls. The Lines were calculated taking the exponential part of the model for each ball after refining the initial parameters of the motion and γ, the lines represent the maximum amplitude of the motion at each instant. It Is clear from the graph that each ball had a different γ.

The Plots of the collected data for each ball and the evolution of the maximum amplitude according to the calculated model for each ball are shown below:

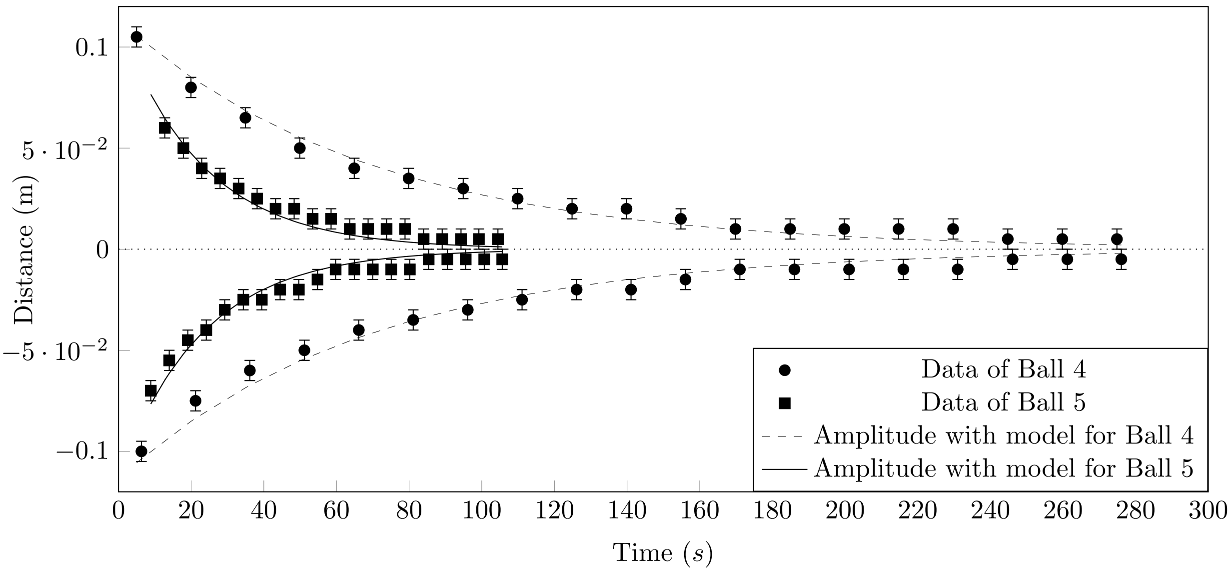

Fig. 5: Graph of distance vs time of the last two balls. The Lines were calculated taking the exponential part of the model for each ball after refining the initial parameters of the motion and γ, the lines represent the maximum amplitude of the motion at each instant. It Is clear from the graph that each ball had a different γ.

from figures **Fig. ??** and **Fig. ??** it is clear that γ is different for each ball. As the value of gamma is different for each ball, it was feasible to conclude that the motion itself is mass dependent, such dependence is not observed in the first periods (**Fig. ??**). If the match between the data and the model was perfect it would be expected to see all the data points in the line of the model as the collected data correspond to the maximum amplitude at that point.

The error in gamma was found using the usual method[5] to determine errors in values calculated via the minimisation of ChiSq.

The method to find the uncertainty on a parameter (P) found by minimising ChiSq using solver consists on varying the parameter by a small amount about the optimum value and then determining the new ChiSq via solver but varying all the parameters but the P. This allows to see how ChiSq changes under small variations of the Parameter around the optimum value. The uncertainty on P is given by the difference between the optimum value of the parameter minus the the value where ChiSq is exactly one unit more than the optimum value.

After refining all the parameters of the motion of each data set using solver to minimise ChiSq the final values of γ obtained for each Motion and the ChiSq that they yield were:

-	Ball 1	Ball 2	Ball 3	Ball 4	Ball 5
$\gamma(s^{-1})$	$1.61(8) \cdot 10^{-3}$	$2.74(7) \cdot 10^{-3}$	$5.47(18) \cdot 10^{-3}$	$1.45(5) \cdot 10^{-2}$	$4.34(16) \cdot 10^{-2}$
ChiSq	32,92	49,39	34,25	41,11	75,60

Table 3: Table that displays the final results of γ and ChiSq

3.3 Analysis on γ

While γ depends on the mass of each ball, the Air Resistance parameter b does not depend on the mass of the balls, just on the geometry of the object moving through the fluid . Therefore, as the geometrical properties are maintained from ball to ball (same radius of the outer polystyrene ball) the air resistance parameter should be the same for all the balls. b relates with gamma and the mass by the equation:

$$\gamma = \frac{b}{2m}$$

This equation was linearised by taking $X = \frac{1}{2m}$ and $Y = \gamma$. Using the obtained values of gamma for each mass, a graph of γ vs $\frac{1}{2m}$ was generated. LINEST was used to generate the best fitting line.

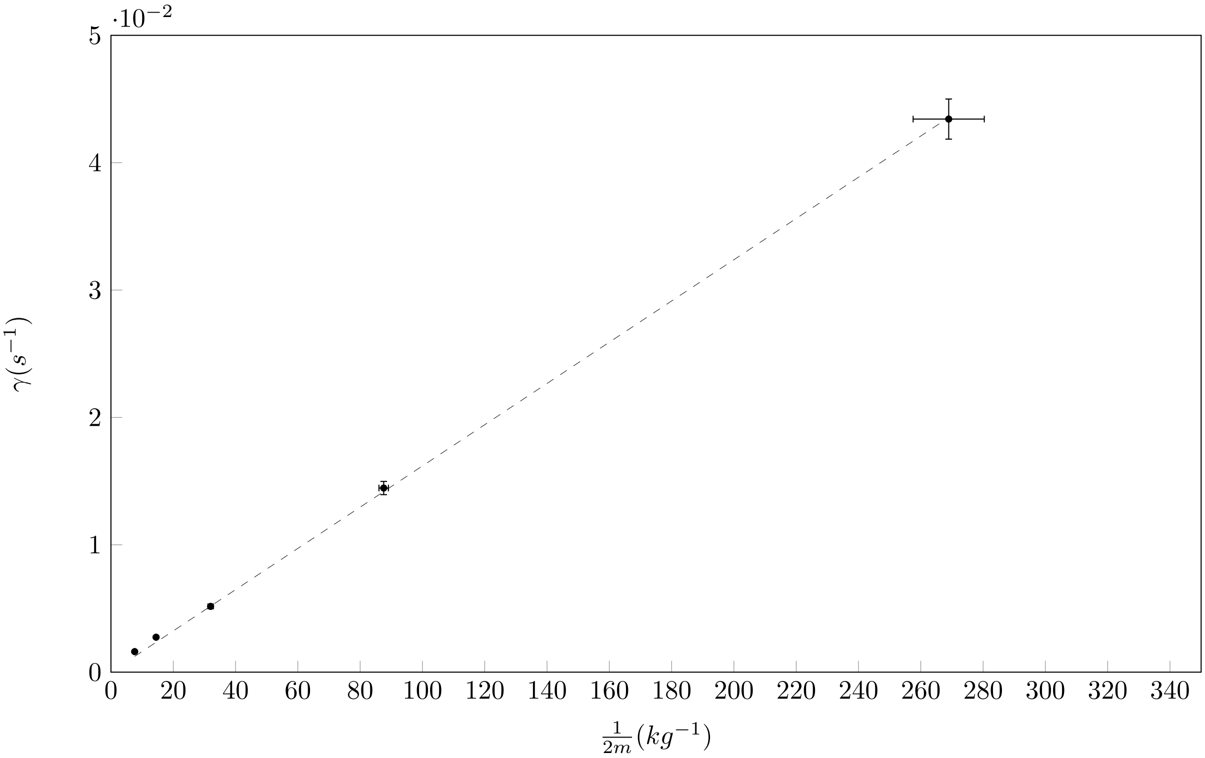

Fig. 6: Graph of the γ vs $\frac{1}{2m}$ The dashed line is the best-fitting line calculated using LINEST in excel (non-weighted linear fit). The slope of the best fitting line corresponds to the damping parameter b, its uncertainty was also calculated through LINEST

The slope of the line determined by LINEST[4] corresponds to the Air Resistance Parameter b, which is equal to 0,0001620(12)$\frac{kg}{s}$. The uncertainty on b was given by LINEST. As the data followed a linear patter it is feasible to conclude that indeed b is a constant for all the balls.

3.4 Effect of γ on ω

The given model predicts that the angular frequency of the motion depends on γ (and therefore on the mass) and on ω_0. This dependence is expressed by the equation:

$$\omega = \sqrt{\omega_0^2 - \gamma^2}$$

It can be seen from the equation that if γ is small compared to ω_0, then the effect of γ in ω will be negligible as the square makes the contribution of γ really small.

[4]Non-weighted linear fitting

To verify that the impact of γ was small in the frequency of the motion ChiSq was calculated after changing ω by ω_0 in the model. The results are shown in the table below:

-	Ball 1	Ball2 2	Ball 3	Ball 4	Ball 5
ChiSq (Before)	32,92	49,39	34,25	41,11	75,60
ChiSq (After)	32,92	49,39	34,25	41,11	75,65

Table 4: Shows the effect on ChiSq of changing w by ω_0 on the model

It can be seen that the change of ω by ω_0 does not affect by much the values of ChiSq[5], so it is feasible to conclude that γ does not have a huge effect on the frequency of the motion.

4 Discussion

4.1 Short Data

All the residuals for the short data look similar after refining the parameters of the model, the residuals for **Fig. ??** have been left as an example:

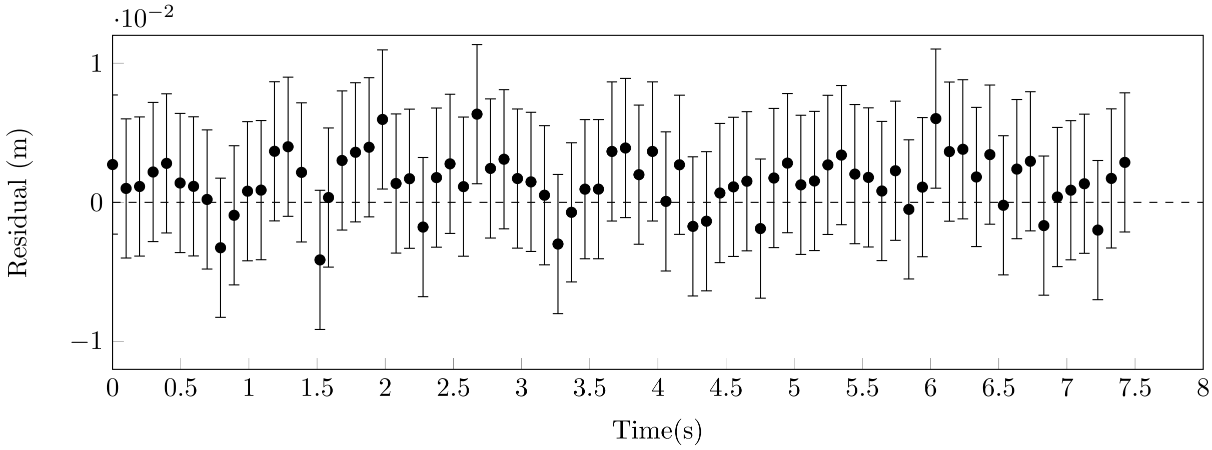

Fig. 7: Plot of the residuals for the data of the 3 first oscillations of the Ball 1. The uncertainties on each data point were $\pm 0.005m$. More than two third of the error bars cross the x-axis so the uncertainties were overestimated

The absence of a clear trend in the residuals and the low values of chiSq for the first 4 Balls (Which ranged from 19 to 56) indicated that the model was a good fit for the data, and that damping had little effect on the first periods of the motion for each ball. However, for some balls (like Ball 1) the number of points above the zero line was clearly higher than the number of below that line, this suggested that the model for the motion of the ball might be benefited by adding an offset (δ). This was added for all the balls in the study of the long range data as another free parameter to be optimized by Solver.
The only interesting case were the residuals for the lightest ball (Ball 5), it was clear from **Fig. ??** than the model did not provide a good fit for the data if γ was set to zero. The residuals for this case are shown below.

[5] As a matter of fact the change in 4/5 balls is less than 0.01 so it does not appear represented on the table

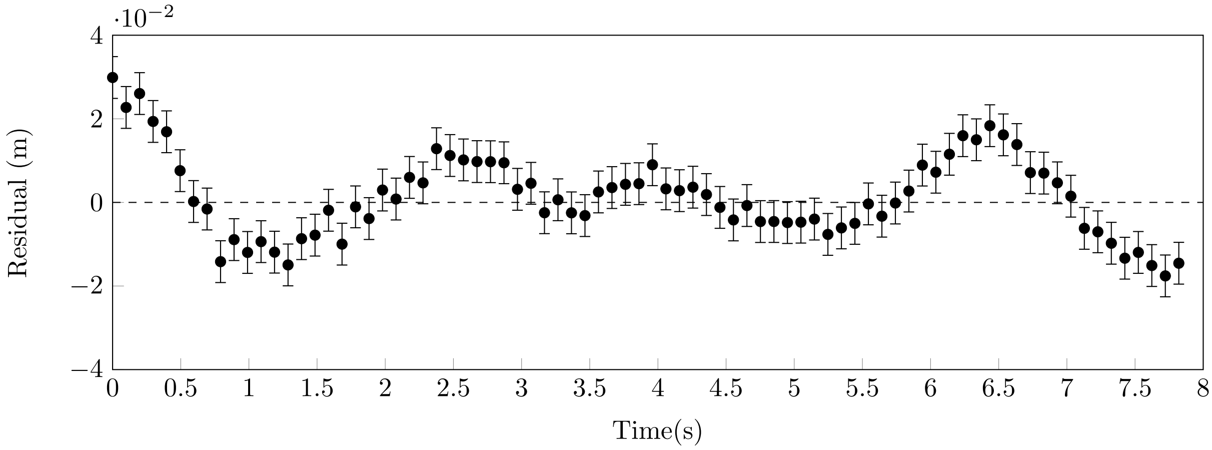

Fig. 8: Plot of the residuals for the data of the 3 first oscillations of the Ball 5. The uncertainties on each data point were $\pm 0.005m$. There is a clear sinusoidal trend on the data, which suggests that the model is not appropriate

Taking into account the Plot of **Fig. ??**, the trend on the Residuals from **Fig. ??** and the high value of ChiSq (~ 300) for this ball, it is feasible to conclude that the model with γ set to zero does not give a good fit for the data[6] of the last ball. This was expected as the damping on this ball is noticeable even on the first periods of the motion.

4.2 Long Data

The residuals plot for each motion looked similar and they are all plotted below:

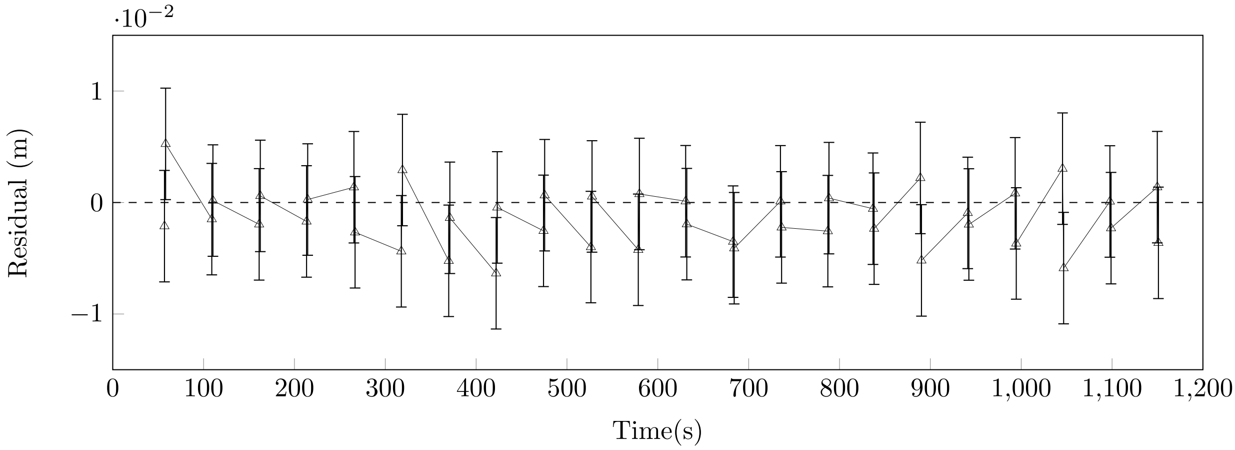

Fig. 9: Plot of the residuals for the long data of the Ball 1. The uncertainties on each data point were $\pm 0.005m$. More than two third of the error bars cross the x-axis so the uncertainties were overestimated. The line is used to help to detect a pattern in the residuals, however none is seen

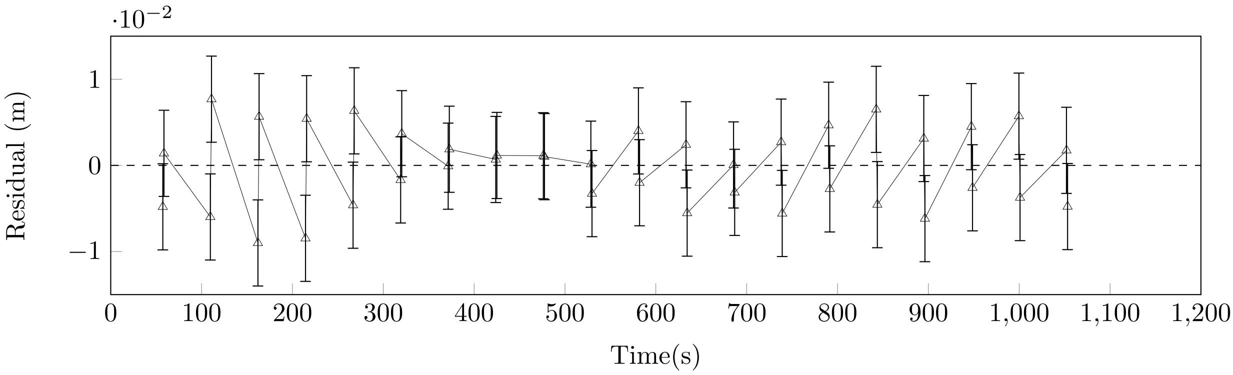

Fig. 10: Plot of the residuals for the long data of the Ball 2. The uncertainties on each data point were $\pm 0.005m$. More than two third of the error bars cross the x-axis so the uncertainties were overestimated.No pattern is detected and they appear to scatter randomly. Also, the number of points above and below the x axis is roughly equal

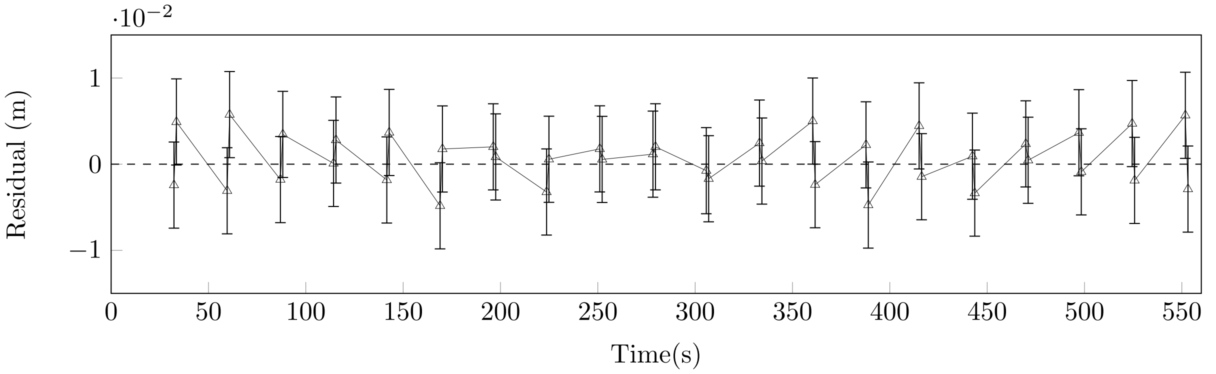

Fig. 11: Plot of the residuals for the long data of the Ball 3. The uncertainties on each data point were $\pm 0.005m$. More than two third of the error bars cross the x-axis so the uncertainties were overestimated. No pattern is seen, as they appeat to scatter randomly about the x-axis.

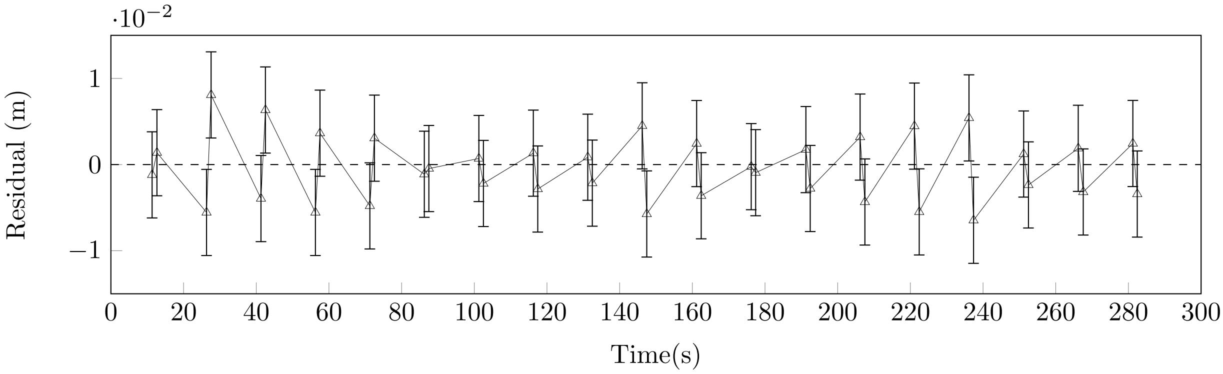

Fig. 12: Plot of the residuals for the long data of the Ball 4. The uncertainties on each data point were $\pm 0.005m$. More than two third of the error bars cross the x-axis so the uncertainties were overestimated. There is no noticeable trend that indicates something wrong with the fit.

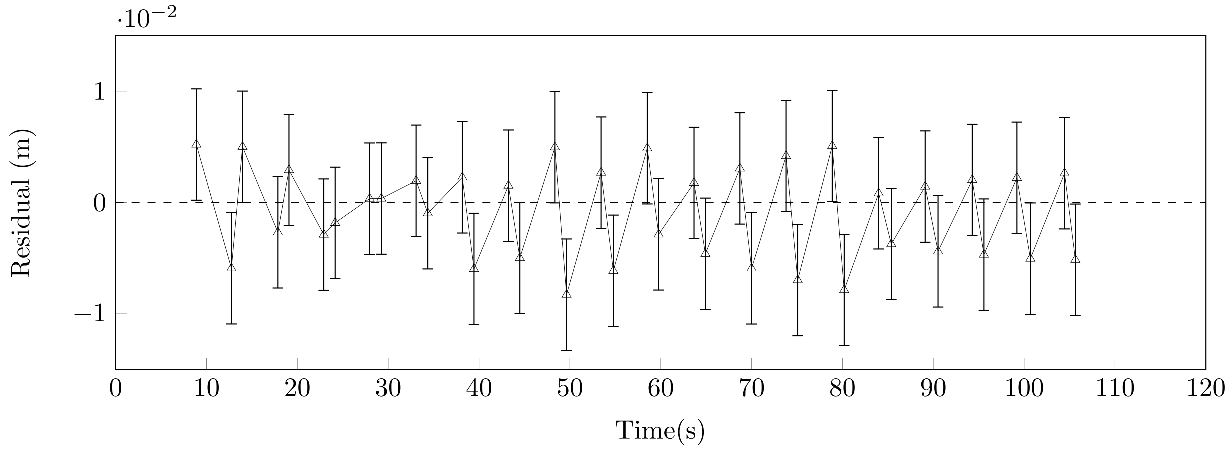

Fig. 13: Plot of the residuals for the long data of the Ball 5. The uncertainties on each data point were $\pm 0.005m$. More than two third of the error bars cross the x-axis so the uncertainties were overestimated. There is a start of a trend on this plot as it has some symmetry about the middle of the plot

In general there were no obvious trends on the difference plots in the first 4 Balls, this combined with the low ChiSq, which ranges from ~ 30 to ~ 45 is sufficient to conclude that the data seems to agree reasonable well with the given model. The difference plot of Ball 5 starts to show a trend on the residuals, which suggest that there might be an issue with either the model or the data[6]. As the model agreed reasonably well with the other balls data, it was more reasonable to infer that there might have been an issue with the data for this ball. Ball 5 was the one that decayed faster, meaning that it spent more time oscillating with a low amplitude. This values were not recorded with perfect accuracy and there was little distinction on these small values. That lack of accuracy might have created a systematic error on the data. However, other explanations regarding the validity of the model itself are discussed on the next section.

4.3 Gamma Analysis

The parameter b was experimentally determined to be $1.620(12) \cdot 10^{-4} \frac{kg}{s}$, this result and its error was calculated using LINEST in excel. The plot of the residuals of the graph that gave this value are plotted below:

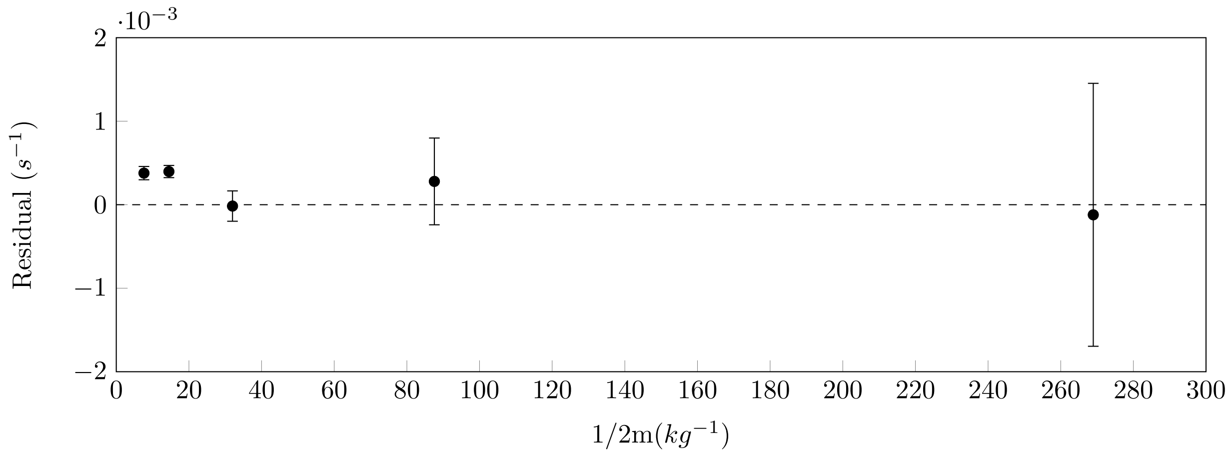

Fig. 14: Plot of the residuals for the analysis of the relation between the mass and γ. No trend is visible at this point, the uncertainties have been considered correctly as $\sim \frac{2}{3}$ of the bars cross the x-axis

It can be seen that there in no apparent trend on the residuals. However, this value is far from the theoretical expectations. The equation used to model the evolution of the motion assumes a linear drag. This type of drag is only possible for small objects moving slowly through a fluid, such that they have a small Reynolds number. If this assumption held then the parameter of the drag would be equal to $6\pi\eta r$ by Stokes's law[3] as the object moving through the fluid is a sphere. Therefore $6\pi\eta r$ should give at least a rough estimate of b, for the case of the fluid being air and the ball having a radius of 0.05 m $6\pi\eta r \sim 10^{-5}$ which is one order of magnitude away from the experimentally determined value.
As the Theoretical value does not match the experimentally determined one it is a good stage to revisit some of the assumptions made for the model. The most important and doubtful assumption on the model of the motion was to assume linear drag without questioning it. The assumption of linear drag would only hold if the Reynolds number of the sphere was less than one (Needed to have linear flow around the object). However, a rough estimate of the Reynolds number gives $R = \frac{\rho D v}{\eta} \sim 700$ for the lowest speed found in the data. Definitely, the Linear Drag assumption does not hold for this case. A more accurate model for drag in this case would be a drag proportional to the square of the velocity, as it is the type of drag that dominates in this range of values of the Reynolds number. However, this change on the drag model would make the problem impossible to solve analytically and numerical methods would be needed. Other studies suggest that an even more appropriate model would be one with drag proportional to both the acceleration and the velocity of the system[7]. However, this approach would also require numerical methods to be studied.

4.4 Effect of γ on ω

It was determined experimentally that the effect of interchanging ω by ω_0 in the model had little effect to the quality of the fit between the data and the mode (The value of ChiSq remained almost constant). This was expected as the values of γ in all instances were small when compared to the calculated values of ω_0. Also, this result is verified by comparing the values of ω and ω_0 through division ($\frac{\omega}{\omega_0}$). The results of this analysis are shown below:

-	Ball 1	Ball 2	Ball 3	Ball 4	Ball 5
$\frac{\omega}{\omega_0}$	0,9999998	0,9999994	0,999998	0,99998	0,9998

The effect of γ on the frequency of the motion is almost negligible, and therefore, it is feasible that the effect of interchanging ω and ω_0 in the model has little effect on ChiSq.

5 Conclusion

Several videos of pendulums with different masses were analysed to obtain an estimate of the decay in each of them. Using a two-step optimisation (to avoid getting stuck in local minimums) procedure the free parameters of the model were refined using Solver Package on excel to perform a weighted non-linear fit. The obtained data were found to agree reasonably well with the proposed model once that the parameters were refined for each ball, with values of ChiSq ranging from ~ 40 to ~ 70 in the worst case. No noticeable trend was seen in the residuals of the first 4 balls, the last one showed the beginning of a pattern which suggested an error on the collection of the data. However, this trend was not very noticeable.
Using the obtained values of γ the key damping parameter b was calculated (as it was a constant for all the balls). However, the value of the experimentally determined damping parameter $b = 1.620(12) \cdot 10^{-4} \frac{kg}{s}$ was one order of magnitude away from the theoretical expectations of the model (Using Stoke's Law). This discrepancy suggested that the model to describe drag was not accurate. Nevertheless, further studies using numerical methods are required to determine a better model for drag. Some alternatives for the linear model would be one proportional to the square of the velocity or proportional to the acceleration and velocity.

References

[1] A. Douglas Davis. THE HARMONIC OSCILLATOR. In *Classical Mechanics*, pages 50–78. Elsevier.

[2] Julia Owen and William Ryu. The effects of linear and quadratic drag on falling spheres: An undergraduate laboratory. *EUROPEAN JOURNAL OF PHYSICS Eur. J. Phys*, 26:1085–1091, 11 2005.

[3] I Eames and C A Klettner. Stokes' and lamb's viscous drag laws. 38(2):025003.

[4] Les Kirkup. *Data analysis for physical scientists: Featuring Excel®*, chapter Appendix 2 - Propagation of uncertainties, page 453–454. Cambridge University Press, second edition, 2012.

[5] Dr. Malcolm McMahon. *Determining the Uncertainties in a Non-linear Fit using Solver.* University of Edinburgh, 1 edition, 2020-2021.

[6] K. Kemmer and S Keller. Nonlinear least-squares data fitting in excel spreadsheets. *Natural Protocol*, (5):267–281.

[7] Robert Salamon, Henryk Kamiński, and Paweł Fritzkowski. Estimation of parameters of various damping models in planar motion of a pendulum. 55(9):1655–1677.