Determination of g and Velocity dependence of Air Resistance

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Contents

Abstract

A value of g was found to be 9.814(4) $\frac{m}{s^2}$ (within 3.5 σ of the literature value) using video analysis software to inspect videos of the fall of a steel ball. Using the recording of the fall of a Polystyrene ball it was found that the Air Resistance on the Polystyrene ball follows a quadratic dependence on the velocity rather than linear.

1 Introduction and theory

On the subject of objects falling, students are usually presented an ideal scenario where the only force acting on such objects is gravity. However, in real-life scenarios, this is not the case. In some cases (For example, in the analysis of the fall of a polystyrene Ball) it is important to consider Drag (Air Resistance). A body immersed in a moving fluid experiences a resultant force due to the interaction between the body and the fluid surrounding it. This force will act in opposite direction to the motion[1]. The presence of this force can be explained by the collisions of the particle that conform the fluid where the object is moving in with the object itself. Those collisions will cause a change in momentum in the falling object, which results in a force.

1.1 Steel Ball

For the case of the fall of a Steel Ball, drag can be ignored and it can be considered a free fall. Therefore, the fall is influenced only by gravity and the mass does not play any role in the acceleration experienced by the object.
The only force acting on the steel box after neglecting Drag is gravity, as the experiment was performed near the surface of the Earth, the value of g is taken to be constant.
Using Newton's second Law (N2) this can be expressed as[1]:

$$\begin{aligned}\overrightarrow{F} &= m\overrightarrow{a}\\ m\overrightarrow{a} &= gm\hat{j}\\ \overrightarrow{a} &= g\hat{j}\\ a_y &= g\end{aligned}$$

This equation can be solved to obtain the position and velocity of the steel ball's centre of mass as a function of time.

$$v(t) = gt \tag{1}$$

$$d(t) = \frac{1}{2}gt^2 \tag{2}$$

1.2 Polystyrene Ball

For the case of a Polystyrene Ball, drag cannot be ignored in the fall. Therefore, the simple analysis for the steel ball, where the only force influencing the fall was gravity, cannot be applied. A careful analysis shows that the mass cannot be canceled. Therefore, there will

[1]For this expressions the vector $\hat{j}$ is positive downwards, pointing to the Earth

be an explicit dependence on the mass in the equations for position and velocity of the Polystyrene Ball. There are two models that have to be considered. Firstly, the case where Drag is proportional to the speed. Secondly, the case where it is proportional to the square of the speed. The dominance of one model over the other will be determined by the Reynolds number [2]. This number is defined as follows:

$$R_e = \frac{\rho l v}{\eta}$$

Where ρ is the density of the fluid, l is characteristic cross-sectional length, v is the speed of the fluid relative to the falling object and η is the dynamic fluid viscosity.

Model 1 The first model to be considered is the case where the drag force is proportional to velocity. This model usually applies for small objects moving at low speeds through a viscous fluid, in this case the Reynolds number will be small [2], and the motion of the fluid around the falling object will be non-turbulent [3].
In this model, the total force acting on the polystyrene ball is given by:

$$\begin{aligned}\overrightarrow{F} &= mg\hat{j} - b|v|\hat{v} \\ \overrightarrow{F} &= mg\hat{j} - bv\hat{j} \\ F_y &= mg - bv\end{aligned}$$

This can be solved to obtain both position and velocity as a function of time:

$$v(t) = v_t(1 - e^{\frac{-t}{\tau}}) \tag{3}$$

$$d(t) = v_t t - v_t \tau (1 - e^{\frac{-t}{\tau}}) \tag{4}$$

Where v_t and τ are given by:

$$\begin{aligned}v_t &= mg/b \\ \tau &= \frac{m}{b}\end{aligned}$$

Model 2 The second model to be considered is the case where the drag force is proportional to the square of the velocity. This model usually applies for objects with larger Reynolds's number[2]. These objects are larger and moving at greater speeds than the ones with small Reynolds's number (Given that the fluid is the same). Therefore, these objects turbulence is more important[3].
In this model, the total force acting on the polystyrene ball is given by:

$$\begin{aligned}\overrightarrow{F} &= mg\hat{j} - k|v|^2\hat{v} \\ \overrightarrow{F} &= mg\hat{j} - kv^2\hat{j} \\ F_y &= mg - kv^2\end{aligned}$$

This can be solved to obtain both position and velocity as a function of time:

$$v(t) = v_t \tanh \frac{t}{\tau} \tag{5}$$

$$d(t) = v_t \ln \left(\cosh \frac{t}{\tau} \right) \tag{6}$$

Where v_t and τ are given in this case by:

$$v_t = \sqrt{\frac{mg}{k}}$$
$$\tau = \sqrt{\frac{m}{gk}}$$

For both models v_t represents the terminal velocity, and τ the characteristic time. This behaviour is confirmed by taking the limit of both expressions for velocity.
In model 1:

$$\lim_{t\to\infty} v_t(1 - e^{\frac{-t}{\tau}}) = v_t(1 - 0) = v_t$$

In model 2:

$$\lim_{t\to\infty} v_t \tanh \frac{t}{\tau} = v_t \text{ (tanh(x) tends to 1 as x teds to infinity)}$$

2 Experimental Methods

2.1 Materials

In this subsection the materials (including software) used to perform the experiment will be presented.

- Large ruler (1.9 meters)
  This ruler was created by joining together A4 squared pages. The separation between marks was of 10 cm.
- Polystyrene Ball
  The radius of the ball was 0.01 m and the mass 0.000 085 7(13) kg
- Steel Ball
  The radius of the ball was 0.01 m and the mass 0.032 640(2) kg
- Phone
  The phone used to record the videos of the falling objects was an IPhone 6, at a frame rate of 240 fps
- ImageJ (Fiji)
  Software used to analyse the videos
- Computer
  Used for data collection and the analysis.

2.2 Experimental Procedure

2.2.1 Experiment Set Up

The experiment required to attach the Long ruler to the wall (to be able the measurement of the distance fallen). The videos were taken some distance away from the wall so that the video could include all the ruler, it was also important to set the phone in a position where the parallax error was minimum. Also, some lights were required to make visible the marks of the ruler on the video. In addition to the ruler, to make the falling process more accurate a cardboard tube was placed at the top of the ruler. Thanks to the tube, the ball could be thrown in a systematic way. It is important to note that the position of the tube must be such that the centre of mass is at the level of the Zeroth mark, this was done to make measurements consistent.

2.2.2 Collection of Data

All the data were collected using the set up described above. After recording three videos for each ball (three videos for the fall of the steel ball and three for the fall of the polystyrene ball), they were exported to ImageJ(Fiji). At this step, there were some technical difficulties (as the original videos could not be imported to ImageJ) so it was decided to use the videos provided by Dr. Malcolm for this experiment.
Once that the videos were in ImageJ, it was time to collect the data required to perform the experiments. ImageJ is a software that allows to analyse videos frame by frame. Using ImageJ it was possible to determine at what frame of the video the centre of mass of the ball reaches the different marks of the ruler, that were 10 cm apart (That is, the frame that it reaches the 0cm mark, the 10cm...). This was found both easier and more precise than measuring the distance traveled in a fixed amount of frames (distance travelled at frame 2, 4,8...), as it is not easy to determine the distance (Only the marks of 10 and 5cm were visible). Some difficulties were found on the process of extracting all the data from the videos.

- In both cases, the balls did not always pass each mark at a determine frame (for example at frame 123), but the passed the mark at some point between two frames.
- The polystyrene ball tended to move to the side of the movie, hindering the measurement.
- The determination of the first frame (When the ball starts falling) was not always clear, this was found to be a bigger problem in the polystyrene ball videos. Due to the importance of the First frame in the following values, this could have created a systematic error in the data.

To solve the first issue, the frames were considered up to $\frac{1}{4}$ of a frame. Allowing measurements of 0.25,0.5,0.75 and 1 frame. The second issue was addressed by also taking into account the position of the shadow of the ball (where possible) to make the measurement more accurate.

3 Results and Analysis

3.1 Steel Ball

Using the collected data, the frames at which the steel ball passes the marks of 0cm, 10cm ...190cm were transformed to the time taken to reach those marks. This was done by dividing by the frame rate (240 fps in this case). The different measurements of the time taken to reach each mark were averaged to get single data set with the best estimate of distance versus time. The error in the distance fallen Δd was taken to be 0.01 m, (The radius of the ball), the error in the time was taken to be the standard deviation $\Delta t = \sigma(t)$. Using this data the distance fallen was plotted against time. The resulting graph is shown in **Fig.** 1.

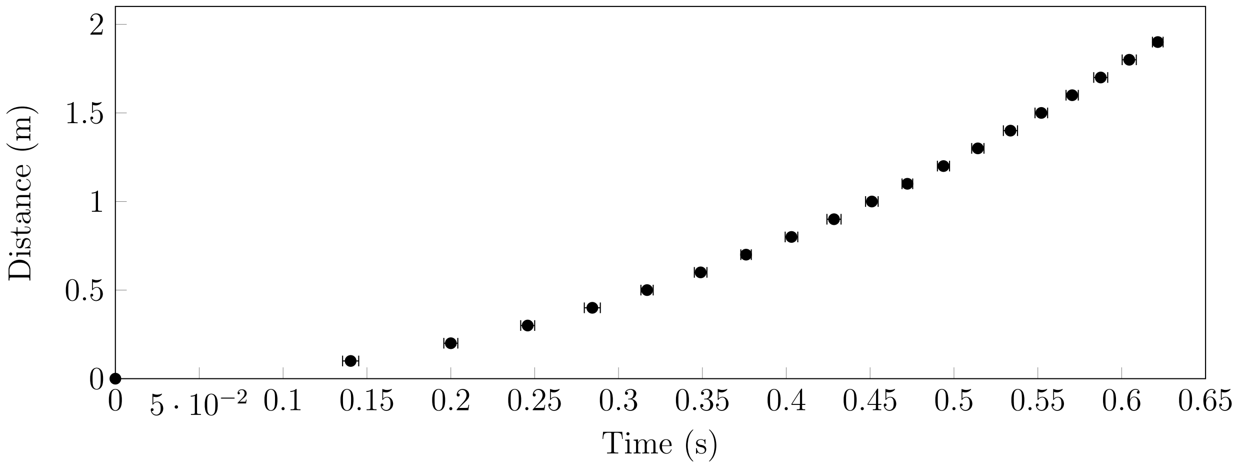

Fig. 1: Graph of the distance fallen vs average time of the steel ball. The uncertainties on each data point are smaller than the symbols used to plot them.

Equation(2) can be linearised, such that Y = distance, and $X = \frac{1}{2}t^2$. The error in $\frac{1}{2}$time^2 was calculated using the usual formula for the propagation of uncertainties [4]. The resulting plot is:

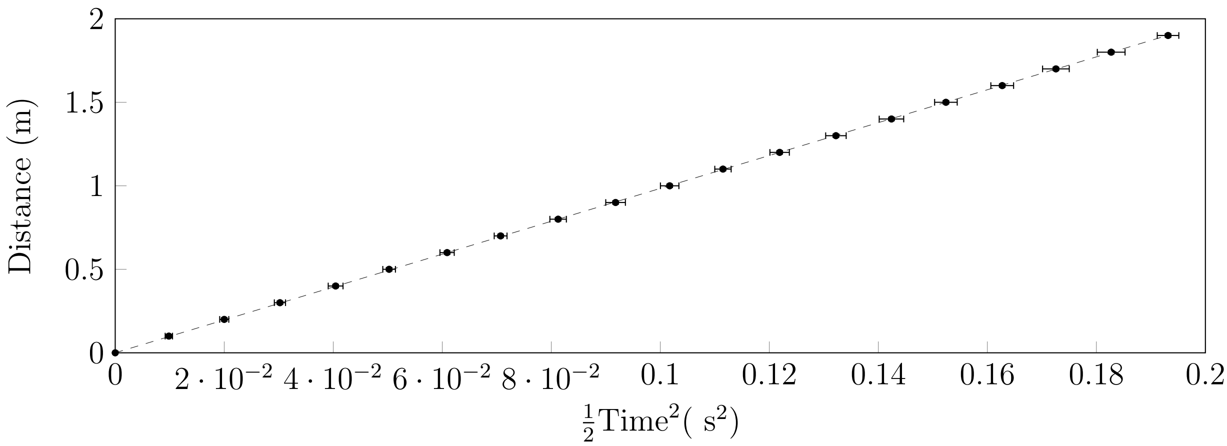

Fig. 2: Graph of the distance vs $\frac{1}{2}time^2$ of the steel ball.The dashed line is the best-fitting line. It was calculated using LINEST in excel.

The line plotted in figure 2 represent the best-fitting line, which was calculated using LINEST function in excel.
As the equations has been linearised, the slope corresponds to the observed value of g. The value of g obtained using LINEST was: 9.844(4) m/s^2.
However, after analysing the data it was noted that the choose of the first frame had a great effect in the value of g.
This dependence was exploited to optimise the obtained value of g. The technique used to optimise the result was minimising ChiSq (obtained by comparing the observed values with the values calculated using the value of g in Edinburgh[2]) by varying the value of the first frame. Solver was used to find the value of the First frame that determined the minimum ChiSq.
This optimisation led to a value of g of

$$g = 9.814(4) m/s^2$$

(After using Linearising the data again and using LINEST to obtain the best fit). The graph of the liniarised data after using solver to minimise chiSq is shown in **Fig.** 3.

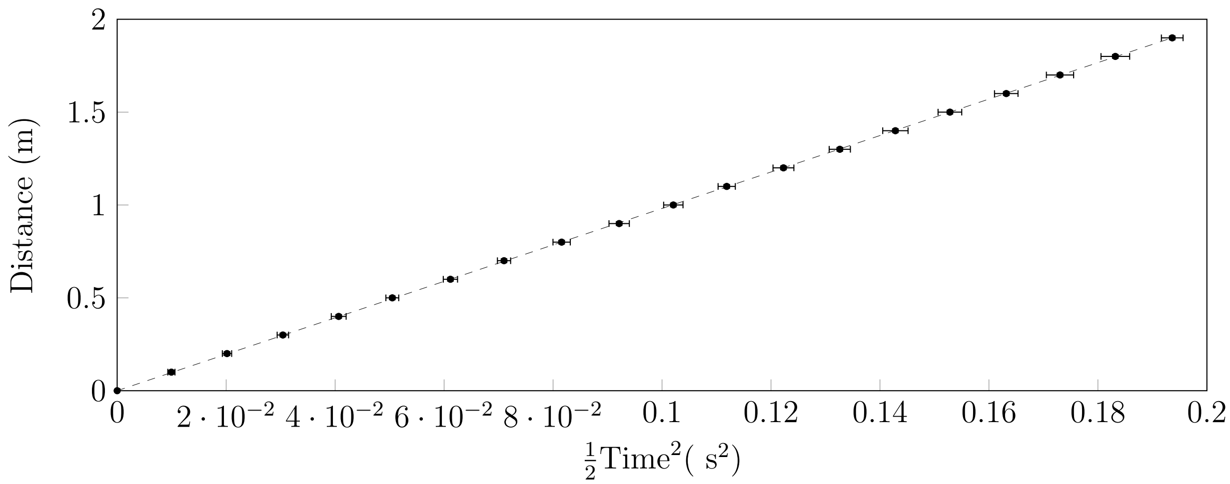

Fig. 3: Graph of the distance vs $\frac{1}{2}time^2$ of the steel ball after minimising ChiSq. The dashed line is the best-fitting line. It was calculated using LINEST (non weighted least-squares method) in excel. The uncertainties on the distance of each data point are smaller than the symbols used to plot them.

3.2 Polystyrene Ball

After collecting all the data, the frames at which the polystyrene ball passes the marks of 0cm, 10cm ...190cm were transformed to the time taken to reach those marks by dividing by the frame rate (240 fps in this case). The different measurements of the time taken to reach each mark were averaged to get single data set with the best estimate of distance versus time. The error in the distance fallen Δd was taken to be 0.01 m, (The radius of the ball), the error in the time was taken to be the standard deviation $\Delta t = \sigma(t)$ in each three

[2]The value given was $g = 9.815791885(7) ms^{-2}$

measurements. Using this data the distance fallen was plotted against time. The resulting graph is shown in figure 3.

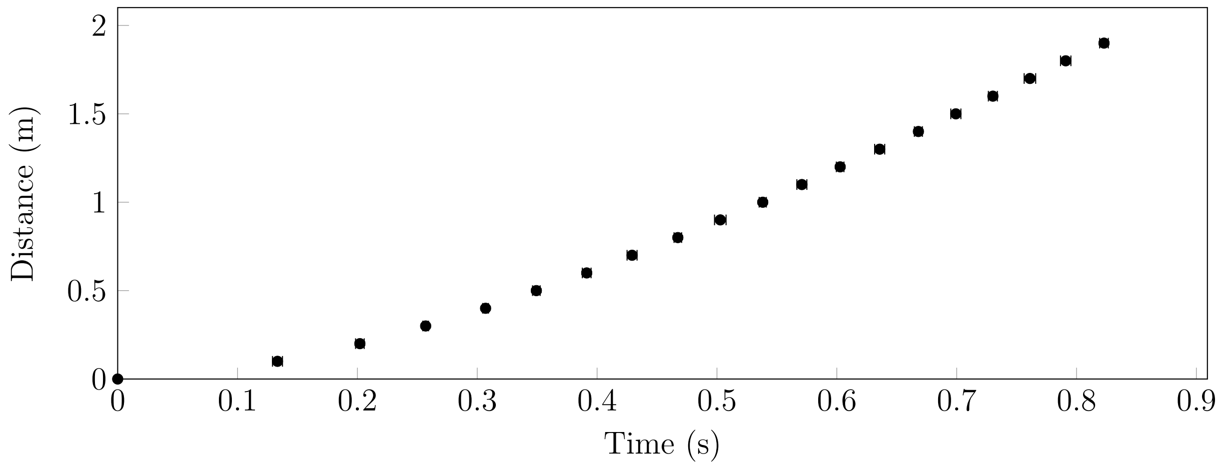

Fig. 4: Plot of the Distance traveled by the Polystyrene Ball against time, The uncertainties on each data point are smaller than the symbols used to plot them.

Determining The Model Solver was used to determine which of the two models result in a better fit, by determining the values of the constants b and k that minimize ChiSq in each model.
The values for b and k found by Solver were:

$$b = 0.0001971(9)\frac{kg}{s}$$
$$k = 0.0000874(6)\frac{kg}{m}$$

The first model, gave a value of 110.48 for ChiSq after the optimisation of b,while the second model gave 36.99 for ChiSq after optimising k.
Fig. 5 shows a comparison between the two models and the observed data.

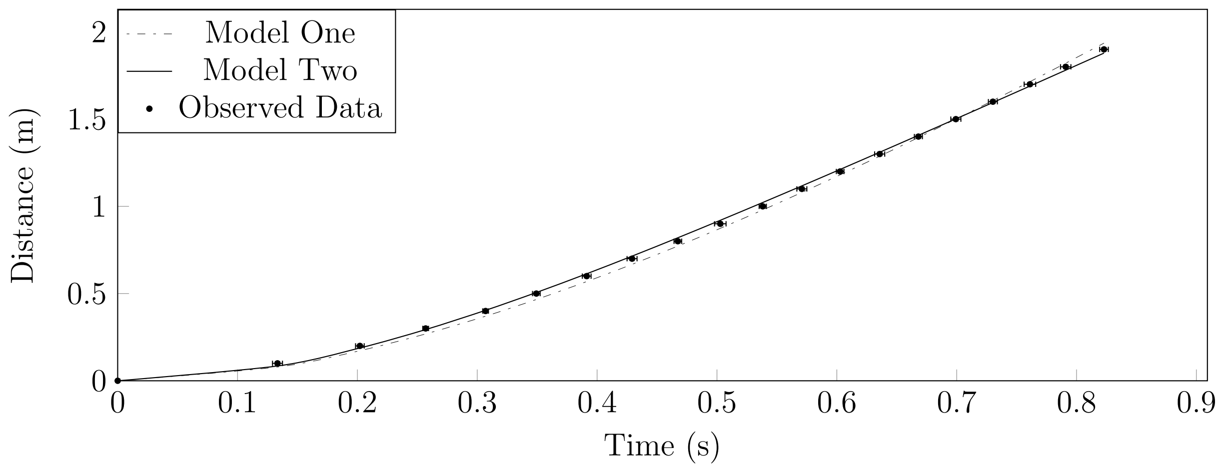

Fig. 5: Plot of Distance against time of the two Models and the data after using solver.

Uncertainties In the Fitting Parameters: The uncertainties in both b and k were determined following the same method [4]. This method is based on calculating the value of ChiSq for values of b and k near the optimum value to determine how rapidly chiSq changes under small variations of the fitting parameters.
The uncertainty on both k and b will is given by the difference between the optimum value of b or k and the value of b or k where ChiSq is exactly one unit more than the optimum value.

Uncertainties in v_t and τ: In both models v_t and τ depend on the fitting parameter (b or k in each case), the mass and g. Therefore, the uncertainties in v_t and τ were calculated by combining the uncertainties in all of those quantities (Δm,Δg and Δb or Δk).
The following method [4] was used to combine all the uncertainties:

Definition. *Let Y be a function of three independent variables:A,B,C. Such that $Y = f(A, B, C)$. then:*

$$\Delta Y = \sqrt{(\Delta A)^2 \left(\frac{\partial f}{\partial A}\right)^2 + (\Delta B)^2 \left(\frac{\partial f}{\partial B}\right)^2 + (\Delta C)^2 \left(\frac{\partial f}{\partial C}\right)^2}$$

Where ΔA, ΔB and ΔC are the uncertainties on A, B, C respectively.

The final values for the terminal velocity and characteristic time (τ) in each model were found to be:

$$v_{t1} = 4.27(6)\frac{m}{s}$$
$$\tau_1 = 0.435(6)s$$
$$v_{t2} = 3.10(3)\frac{m}{s}$$
$$\tau_2 = 0.316(11)s$$

4 Discussion

4.1 Steel Ball

The experimentally determined value of g was: $9.814(4)m/s^2$, after optimising the value of the first frame. This result is both accurate and precise. It is precise as the error on the measurement is small and it is accurate as the value is close to the literature value of g in Edinburgh, more precisely the found value is within 0.02% or 3.5σ of the literature value (which is $g_r = 9.815791885(7)ms^{-2}$). Also, The real value of g is within the error for the observed g which is a good sign.
A plot of the residuals of the Linearised data after optimising the value for the first frame is shown in **Fig.** 6.

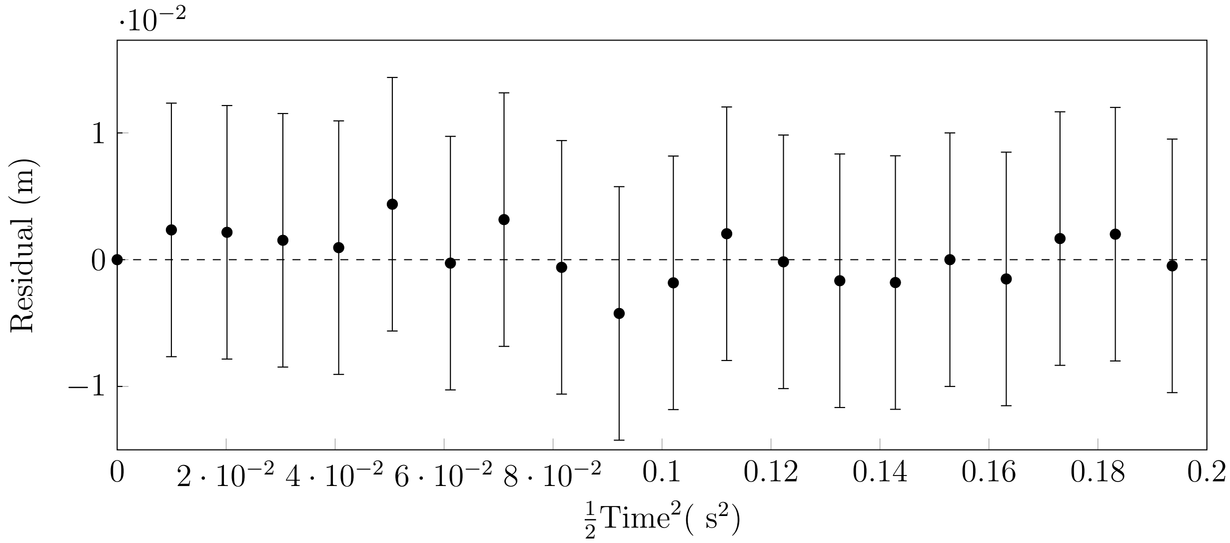

Fig. 6: Plot of the residuals for the Steel Ball. The unceratainties on each data point are $\pm 0.01m$

Fig. 6 shows that the residuals are randomly scattered about the x-axis, which is a good sign. If a trend was found in the distribution of the residuals about the x-axis, that would mean that either there is something wrong with the data or something wrong with the model used to fit the data [5]. However, It is clear than the scatter of the residuals about the x-axis is less than error bars in each data point, as more than $\frac{2}{3}$ of the error bars cross the x-axis. This suggests that the uncertainties in the distance have been overestimated. The overestimation of the uncertainties is because the chosen uncertainty for the distance was the radius of the steel ball, which was the maximum possible uncertainty.
The value of g determined without optimising the first frame with solver, $9.844(4)ms^{-2}$ is also accurate (it has the same uncertainty as the optimised one) and precise (it is within 0.3% of the literature value).

4.2 Polystyrene Ball

The optimised value of ChiSq for model 1 was determined to be 110.48, while the optimised value for ChiSq in model 2 was found to be 36.99. The values of the fitting parameters that gave the minimum value of ChiSq were:

$$b = 0.0001971(9)\frac{kg}{s}$$
$$k = 0.0000874(6)\frac{kg}{m}$$

The value of ChiSq gives an insight of how close are the experimentally observed values to the calculated values using each model. Therefore, a smaller value on ChiSq indicates that the experimentally determined values are closer to the theoretical values. Hence, The best fit is given by the model with less value for ChiSq after optimisation so model 2 gives a better fit. However, it is also important to analyse the residuals plot for both models. A plot for the residuals of model 1 and model 2 are shown in **Fig.** 7 and **Fig.** 8 respectively.

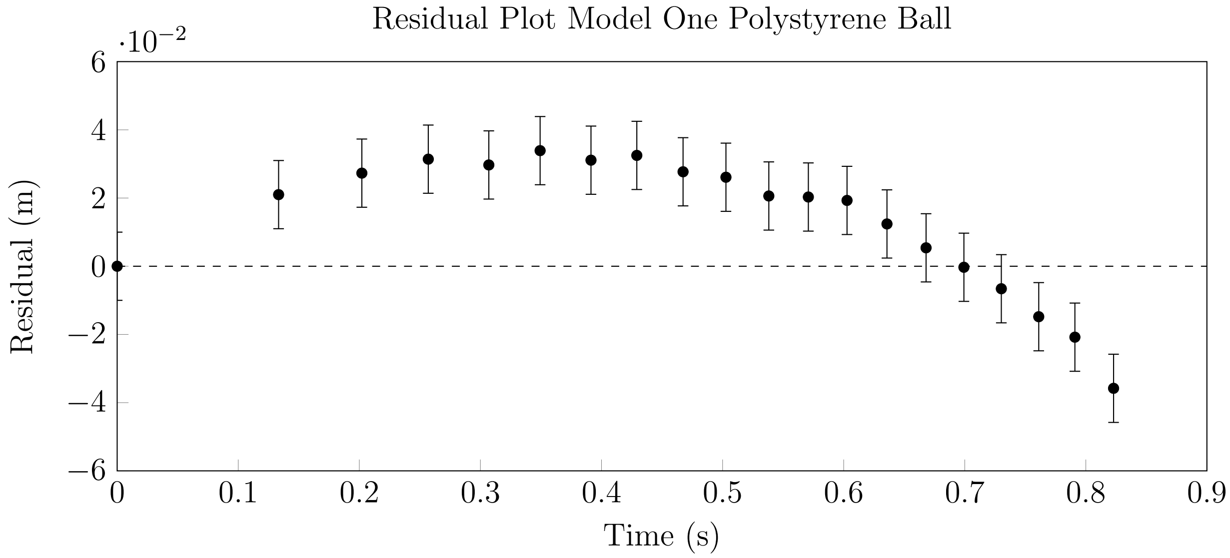

Fig. 7: Plot of the residuals for Model 1. The uncertainties on each data point are ±0.01cm, given by the radius of the polystyrene ball.

It is clear from **Fig.** 7 that the residuals for model one follow a trend. As the value of ChiSq is also less for this model, the presence of a trend can be explained by the fact that the model is just not suitable for the collected data [6].

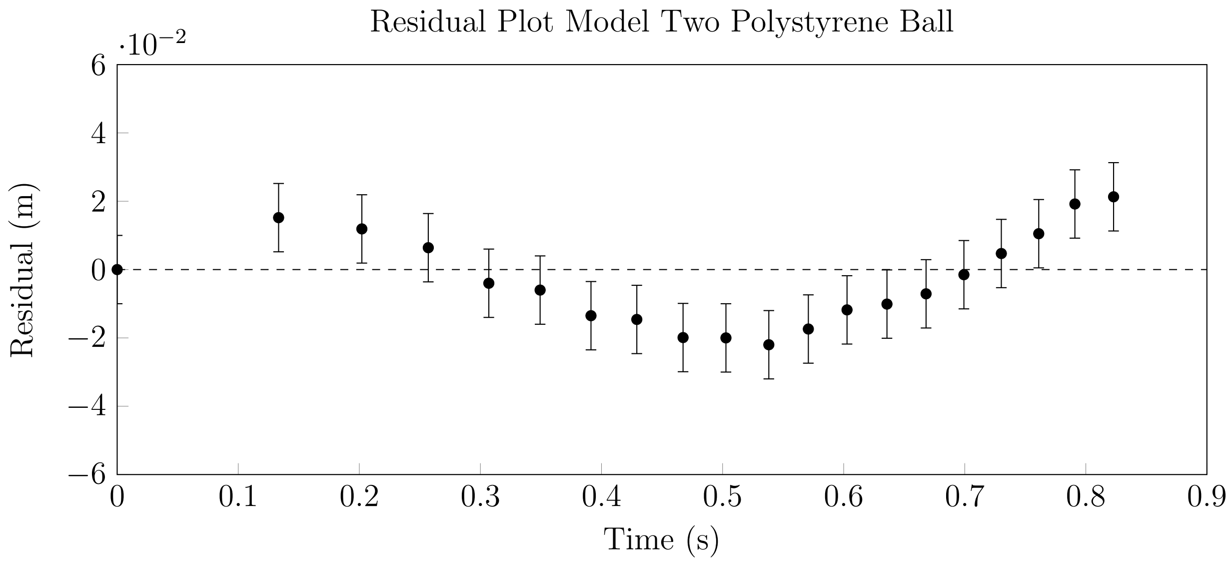

Fig. 8: Plot of the residuals for Model 2. The uncertainties on each data point are ±0.01cm, given by the radius of the polystyrene ball.

From **Fig.** 8 it is clear that the residuals of model 2 do not follow a trend as obvious as the followed by model 1 residuals. However, there is still a trend in the distribution of the residuals about the x-axis. The presence of a trend in the residuals might indicate a systematic error in the data. The most probable error made through the experiment could

be the determination of the first frame (Zero frame), this value affects all the other values and it is a probable source of error. However, the trend in the residuals for model 2 is not as noticeable as the trend followed by the residuals in model 1 (which can be explained by the model not being suitable for the data)
The lower value of ChiSq is enough to conclude that the model 2 is a better fit for the data collected. Therefore, the data suggests that the polystyrene ball is affected by an air resistance with quadratic dependence on the velocity rather than linear. This in agreement with the theoretical expectations as the Reynolds number of a Polystyrene ball (2 cm of diameters) in air is about 10^3 which is definitely not less than one, and therefore the Air resistance should have a quadratic dependence on velocity rather than linear [2].
In addition, the data plotted in **Fig.** 4 (distance vs time of the polystyrene ball) shows a linear tendency from approximately the second 0.4 onwards. As the graph displays the relationship between space and time, finding a linear relationship between them indicates that the derivative of distance with respect of time (velocity) is constant. Therefore, it is feasible to conclude that the polystyrene ball reaches the terminal velocity before passing the 190 [cm] mark.

5 Conclusion

Several videos of a steel ball falling were used to determine the value of g. With the aid of video analysis software it was analysed how much time it takes the ball to reach marks separated by 10cm. By analysing the relationship between the distance fallen and the time that it takes to the ball to reach it, g was determined to be 9.814(4) $\frac{m}{s^2}$. This is within 0.02% (3.5 sigma) of the literature value in Edinburgh.
After performing the same analysis on the videos of a Polystyrene Ball falling, it was found that the Air resistance acting on the ball has a quadratic dependence on velocity rather than linear. This conclusion is also in agreement with the theoretical expectations, as the Reynolds number of a sphere of such radius moving through air is not small enough to make the Air resistance linear in velocity.

References

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