Neutrino Oscillations

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Abstract

Neutrinos are one of the more elusive particles in the standard model and a reason for this is their incredibly small mass. While initially assumed to be zero, the idea of a 'massless neutrino' was disproved with the discovery of neutrino oscillations; the phenomenon observed where neutrinos transmute between different types or "flavours". Using a quantum mechanical framework, a derivation for neutrino oscillations was formed in this paper. Following from this, the experimental results from Super-Kamiokande and Sadbury National Observatories were presented showing evidence of neutrino oscillations and proof that neutrinos have non-zero mass. The implications of neutrino oscillations, such as the validity of claims made about neutrino speed, as well as the possibility of charge parity symmetry violation were discussed.

1 Introduction

The standard model of particle physics is a widely accepted theory that defines the smallest particles in the universe as well as three of the four fundamental forces that act upon them, these particles are known as *elementary particles*. Neutrinos belong to a sub-category of elementary particles called Leptons and there are 3 different types (or flavours): the *Electron Neutrino*, the *Muon Neutrino* and the *Tau Neutrino*. Neutrinos are interesting for being one of the most abundant particles in the Universe, with roughly a trillion passing through our bodies each second from the sun alone [1]. Due to their extremely high abundance (approximately $O(10^{87})$ in the observable universe) [2], it is believed that if they have mass they would affect the mass density of the observable universe. The amount of mass that some types of neutrinos do have is still fairly uncertain and at a certain point it was believed that they were all massless. This massless belief lead to the assumption that neutrinos travelled at the speed of light and then in 2011 when the OPERA Collaboration claimed to have detected neutrinos travelling faster than light [3], the upper bound of the speed of neutrinos was brought under scrutiny. In this paper we will delve into the phenomenon of *neutrino oscillations* and the implications it has on the massless assumption of neutrinos.

1.1 A Brief History

In 1930 Wolfgang Pauli wrote a letter addressed to a nuclear conference in Tübingen in which he proposed the existence of an undiscovered particle after studying the conservation of energy and momentum within the process of β-decay [4]. We now know that β-decay is the process by which a neutron becomes a proton, emitting an electron and neutrino in the process but at the time of Pauli's proposal, neutrons were 2 years away from being discovered. What was known about β-decay at the time was that under this process, an atomic nucleus would gain a positive charge and emit an electron. The problem with this limited understanding is that when experimentally measuring the energy in a calorimetric β-decay experiment (considering the energy of the electron and the newly formed proton), the experimentally measured energy is smaller than the total energy that should be released. Pauli proposed this could only be possible if there was a neutral particle being emitted along with the electron, that could not be detected in this experiment. Due to the fact that the neutrino is neutrally charged, has almost no interactions with matter and close to massless [1], detecting a neutrino was incredibly challenging but in 1953 Frederick Reines and Clyde Cowan proposed an experiment to detect one which won the 1995 Nobel Prize [5].

2 Neutrino Oscillations

2.1 Quantum Mechanics

Neutrinos follow the rules of Quantum Mechanics, and neutrino oscillations are a purely quantum effect; hence it is important to summarize some of the fundamental rules of Quantum mechanics [6; 7].

- The state of a quantum object is represented by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$
- Each observable O is associated with a hermitian linear operator $\hat{O}$ that acts on $\mathcal{H}$, and all the possible values of the observable are the operator's eigenvalues.
- The set of eigenstates $\{|n\rangle\}$ with eigenvalue n of a Hermitian operator $\hat{O}$ represents a basis of $\mathcal{H}$. Any state $|\psi\rangle$ in $\mathcal{H}$ can be expressed as a linear superposition of that set of eigenstates. The probability of measuring

the value n when measuring the observable O is given by $| \langle n|\psi\rangle |^2$

- The time evolution of a state $|\psi\rangle$ is given by the Schrodinger's equation (SE)

2.2 Definition of Neutrino Oscillations

When a neutrino is generated from the decay of other particles, it is generated with a particular flavour. However, it has been observed that neutrinos produced in a specific flavour can be detected as having a different flavour when measured after travelling through vacuum. This effect is known as neutrino oscillations [8].

2.3 Neutrino States

Both mass and flavour of neutrinos are intrinsic properties of these particles, and hence, they are independent of the spatial properties of these particles. Thus, the flavour and mass of the neutrinos are described in a different Hilbert Space than the spatial one[1]. However, flavour eigenstates are not the same as mass eigenstates in this mass-flavour Hilbert space. One can express each of the flavour eigenstates as a linear combination of the mass eigenstates [9] as they form a basis in this space:

$$|v_\alpha\rangle = \sum_k U^*_{\alpha k} |v_k\rangle \tag{1}$$

In this equation $|v_\alpha\rangle$ represents a flavuor eigenstate (state with definite flavuor) where α can take the values: $\{e, \mu, \tau\}$. On the other hand, $|v_k\rangle$ are eigenstates of the mass (states with definite mass) and $U^*_{\alpha k}$ are just the coefficients of the expansion (These coefficients are represented by a unitary matrix to preserve the inner product) where k can take the values $\{1, 2, 3\}$. Note that mass eigenstates are denoted by $|v_{\text{number}}\rangle$ and the flavour eigenstates by $|v_{\text{symbol}}\rangle$.
On the other hand, it is also possible to invert Equation 1 to express the mass eigenstates in terms of flavour eigenstates:

$$|v_k\rangle = \sum_\alpha U_{\alpha k} |v_\alpha\rangle \tag{2}$$

Note that these states do not represent all the properties of the neutrino as they lack the spatial part. Hence, the state that describes the particle lays on the Hilbert space given by the tensor product between the two Hilbert spaces.

$$\mathcal{H}_{\text{Complete}} = \mathcal{H}_{\text{Mass-Flavour}} \otimes \mathcal{H}_{Space} \tag{3}$$

Another important set of states concerning neutrino oscillations are the energy eigenstates (eigenstates of the Hamiltonian). The neutrino in a vacuum can be considered a free particle so the eigenstates of the Hamiltonian will be given by the tensor product of the eigenstates of the mass and the eigenstates of the momentum [10]:

$$|E\rangle = |v_k\rangle |p\rangle = |v_k, p\rangle \tag{4}$$

Which are the states with defined momentum and mass. These states are important as the time evolution of these states is given by the multiplication of the state by[2] e^{-Eit}. As the Hamiltonian is a hermitian operator, any vector of $\mathcal{H}_{\text{Complete}}$ can be expressed as a linear superposition of the energy eigenstates.

2.4 Derivation of Neutrino Oscillations

Let us consider a neutrino that is generated by a decay at position and time (x_P, t_P) and finally detected at position and time (x_D, t_D).
Due to conservation laws the generated neutrino will be in a state ($|\psi^P_\alpha\rangle$) which corresponds with an eigenstate of flavour α. $|\psi^P_\alpha\rangle$ encapsulates all the information about the particle and it lays in $\mathcal{H}_{\text{Complete}}$. This state can be decomposed in terms of the energy eigenstates described above.

$$|\psi^P_\alpha\rangle = \sum_k U^*_{\alpha k} |v_k\rangle |p_k\rangle \tag{5}$$

The state $|\psi^P_\alpha\rangle$ is expanded in terms of the energy eigenstates which have mass m_k and momentum p_k, the coefficients of the expansion match the ones described in Equation 1 as the resulting state must also be a flavour eigenstate. Note that the expansion of the state only includes three terms; this is because, for each allowed value of m_k (which there are three), there is only one possible value of p_k allowed by momentum conservation of the decay process in which the neutrino was generated [10].
In many derivations [9] it is assumed that the momentum corresponding to each mass eigenstate is the same for all mass eigenstates, and while this might be a good approximation for some decays, it is not correct for others [11; 10].
It is useful to express the state in the energy basis as the time evolution of the state is easy to compute. As described in the previous section the time evolution of the superposed state is given by multiplying each factor by $e^{-E_k i(t-t_P)}$[3], where E_k is the energy of the eigenstate $|m_k\rangle |p_k\rangle$ and its value is given by $E_k = \sqrt{m_k^2 + p_k^2}$

$$|\psi^P_\alpha(t)\rangle = \sum_k U^*_{\alpha k} e^{-E_k i(t-t_P)} |v_k\rangle |p_k\rangle \tag{6}$$

At this point it is useful to take the inner product with $\langle x|$, by doing this the resulting state lays in $\mathcal{H}_{\text{Mass-Flavour}}$ and it is a function of both x and t.

$$\langle x|\psi^P_\alpha(t)\rangle = |\psi^P_\alpha(x,t)\rangle = \sum_k U^*_{\alpha k} e^{ip_k(x-x_P)-E_k i(t-t_P)} |v_k\rangle \tag{7}$$

[1] The spatial Hilbert space is the one where the momentum and position operator act
[2] Note that natural units will be assumed throughout the whole paper, that is: $c = \hbar = 1$
[3] We use $t - t_P$ instead of t to account for the fact that the particle is produced at time t_P

If Equation 7 was evaluated at (x_P, t_P), the resulting state would be the flavour state $|v_\alpha\rangle$ as described in Equation 1. This is to be expected as we know that the neutrino is in state of definite flavour when it is generated.

Equation 2 can be used to express the mass eigenstates as the superposition of flavour eigenstates and get an expression in terms of the latter.

$$\left|\psi_\alpha^P(x,t)\right\rangle = \sum_\beta \sum_k U_{\beta k} U^*_{\alpha k} e^{ip_k(x-x_P)-E_k i(t-t_P)} |v_\beta\rangle \tag{8}$$

If the state was measured at (x_D, t_D), then the probability of measuring flavour γ would be given by the square of the inner product $\left\langle v_\gamma \middle| \psi_\alpha^P(x_D, t_D)\right\rangle = \mathcal{A}_{v_\alpha \to v_\gamma}$ [10]:

$$\mathcal{A}_{v_\alpha \to v_\gamma} = \sum_k U_{\gamma k} U^*_{\alpha k} e^{ip_k L - E_k iT} \tag{9}$$

Where $L = (x_D - x_P)$ and $T = (t_D - t_P)$

Hence, the overall probability of measuring a γ flavour neutrino starting with an α flavour neutrino produced in some particle decay is given by the modulus squared of Equation 9:

$$P_{v_\alpha \to v_\gamma} = \sum_{k,j} U_{\gamma k} U^*_{\alpha k} U^*_{\gamma j} U_{\alpha j} e^{i(p_k - p_j)L - (E_k - E_j)iT} \tag{10}$$

At this stage it can already be seen that the probability of measuring a neutrino of flavour γ when it was initially an α neutrino is not zero and it oscillates as the particle travels in space and time.

2.5 Final Results

While Equation 10 already shows the key features of neutrino oscillations it can be simplified even further using relativistic approximations [10]. These approximations are valid as the propagation velocity of neutrinos is close to the speed of light and because the mass eigenvalues of neutrinos are believed to be small. For further information about the approximations and about the simplification process read [10].

These approximations leads to the following expression for the probability:

$$P_{v_\alpha \to v_\gamma} = \sum_{k,j} U_{\gamma k} U^*_{\alpha k} U^*_{\gamma j} U_{\alpha j} e^{-iL\frac{\Delta m^2_{kj}}{2E}} \tag{11}$$

The resulting probability will not always be zero, but a sum of oscillating terms which depend on the distance L between the source and the detector. Hence, a flavour different than the original one can be measured at the detector. This equation encapsulates the effect described in the introduction.

2.6 Two State system

In many instances, a good approximation to consider is a system in which there are only two possible neutrino flavours $(|v_e\rangle, |v_\mu\rangle)$ and two possible neutrino mass states $(|v_1\rangle, |v_2\rangle)$. In such a case, the matrix U can be represented as a 2D rotation parametrized by an angle θ. In this system the probability described in Equation 11 can be simplified to [12]:

$$P_{\nu_\alpha \to \nu_\gamma} = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right) \tag{12}$$

if and only if $\alpha \neq \gamma$.

3 Experimental Evidence

3.1 Background

The probability function for a two flavour system derived above in Equation 12 serves as an adequate approximation for the following neutrino oscillations experiments [13]; many detectors cannot detect all three neutrino flavours. In Equation 12 there are two parameters that experimentalists can vary: the 'baseline' L which indicates the distance from the source to the detector and E the energy of the neutrino.

An important aim of neutrino oscillation experiments is to determine a value for Δm^2, the mass square differences between two eigenstates.

There are two types of possible experiments. One where we look at the probability of decay; that is, we start with a 'pure' beam of neutrino flavour and measure the survival rate. The second type of experiment is an 'appearance' experiment where we start with a pure beam of neutrino flavour and look at the proportion of new neutrino flavours that have been detected. In the following examples, we will look at a brief history starting from how neutrino oscillations were first experimentally detected to their confirmation several decades later.

3.2 Solar Neutrinos

The discovery of the effects of neutrino oscillations spans back to 1965, when the quest for the accurate detection of solar neutrinos began with the Homestake Experiment [14]. The goal of the experiment was to precisely measure the flux of electron neutrinos from the sun. Following several years of data collection, the final results were peculiar and surprising; a mysterious deficit of solar neutrinos was present – the rate at which electron neutrinos were emitted was only a third of the theoretical expectation [14]. At the time, there was some scrutiny surrounding this result as it went against the well-accepted Standard Solar Model (SSM). This strange result led to further investigation into the discrepancy between the theoretical predictions and experimental evidence and was known as the *Solar Neutrino Problem*.

The resolution to this problem came in the early 2000's from the Sudbury National Observatories (SNO), which successfully proved that solar electron neutrinos were oscillating into different flavours. The SNO relies on a deuteron reaction to create a detectable charged lepton from a neutrino interaction. The deuteron nucleus requires a very small energy to break apart [15], and therefore all specific neutrino flavours (tau, muon or electron) could be detected, unlike the Homestake experiment, which was unable to distinguish between neutrino flavours [14]. The SNO results for solar neutrino fluxes were remarkable [16]:

$$\phi(\nu_e) = 1.75 \pm 0.07 \times 10^6 cm^{-2} s^{-1} \quad (13)$$

$$\phi(\nu_{tot}) = \phi(\nu_e + \nu_\tau + \nu_\mu) = 5.44 \pm 0.99 \times 10^6 cm^{-2} s^{-1} \quad (14)$$

The total flux of neutrinos $\phi(\nu_{tot})$ is in very good agreement with the Standard Solar Model which predicts a flux of $5.05 \times 10^6 cm^{-2} s^{-1}$. Additionally, the number of detected electron neutrinos ν_e is $\approx 1/3$ of the total flux, showing that $\approx 2/3$ of solar ν_e neutrinos oscillate to ν_τ and ν_μ flavours.

3.3 Atmospheric Neutrinos

In 1998, the Japanese neutrino observatory Super-Kamiokande ran a crucial experiment which ultimately first confirmed the existence of neutrino oscillations. On earth, there are constant bombardments of cosmic rays which collide with particles in our atmosphere [17]. Atmospheric neutrinos are produced from the decay of these particles after collision. Super-Kamiokande considered $\frac{\nu_\mu}{\nu_e}$, the ratio of atmospheric muon neutrinos to electron neutrinos and compared it to the theoretical expectation [17].
Naturally, any detector will be located close to the earth's surface. Super-Kamiokande is located 1km underground [18]; neutrinos can easily penetrate that deeply as they don't interact with the matter/rock above. Super-Kamiokande was built with the ability to reconstruct the direction of neutrinos thanks to scattering methods. Atmospherically, Super-Kamiokande can detect neutrinos from 15km away if they are arriving from above (downwards) and equally detect neutrinos that are travelling 13000km [17] from the other side of the earth (upwards) by varying the zenith angle shown in Figure 1.

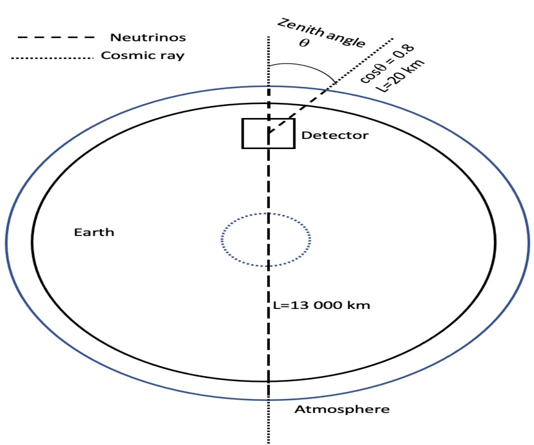

Fig. 1: Diagram of Super-Kamkiokande neutrino zenith dependence

If one were to measure the flux of both upward and downward travelling neutrinos, we would expect the same flux for neutrinos going up and neutrinos going down. The Super-Kamiokande experiment observed that the number of muon neutrinos going up to the detector was only half of the number of muon neutrinos going down [17]. Neutrino oscillations provide a clear explanation for the discrepancy in these results. The atmospheric neutrinos start their journey as defined flavour states, but having travelled different distances, the phase between the mass states will have been altered causing different probabilities of capturing different neutrino flavours at the detector.

3.4 Implications

The main implication from the aforementioned experiments is that they provide concrete experimental evidence for neutrino oscillations and provide detailed measurements of mass square differences between neutrino mass states. For neutrino oscillations to occur, neutrinos must have non-zero mass. This can be seen by Equation 12: if $\Delta m^2 = 0$, there would be no neutrino oscillations at all as the probability $P_{\nu_\alpha \to \nu_\gamma}$ of detecting a new flavour would be zero. Thus, showing that neutrino oscillations do indeed occur proves that neutrinos have mass. These pivotal experiments have helped verify that fact. The Nobel Prize in 2015 was awarded to both Super-Kamiokande and the Sudbury National Observatories for their work on neutrino oscillations [19].

This result can also be utilised to disprove other claims such as the claim that neutrinos can have superluminal velocities. In 2011 the OPERA experiment presented their shocking result that neutrinos can travel at velocities faster than light [3]. The experimental result from Super-Kamiokande that neutrinos have non-zero mass disproves this claim. This is easily shown with special relativity by looking at the following equation for relativistic mass.

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (15)$$

Where m and m_0 are the relativistic and rest mass respectively, c is the speed of light and v is the speed of the object. As can be seen by Equation 15, if we assume that $v \to c$ and a non-zero initial neutrino mass, the neutrino mass and energy would need to tend to infinity to reach that condition. This is impossible with our current understanding of the laws of Physics.

4 Further Studies

The existence of neutrino oscillations raises several questions fundamental to our understanding of particle physics and the universe.

One of the open questions regarding neutrino oscillations is the mass ordering of the mass eigenvalues. While experiments have shown that Δm^2_{21} [20] is positive experimentally, the sign of Δm^2_{31} is yet unknown [20]. Therefore, there are two possible orderings for the three neutrino masses, the normal hierarchy of $m_1 < m_2 < m_3$ and the inverted hierarchy of $m_3 < m_1 < m_2$. The following diagram represents the two possible orderings:

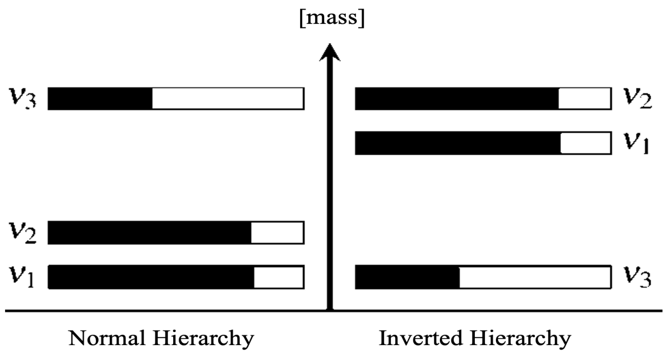

Fig. 2: Diagram showing the two possible hierarchies of mass ordering adapted from [21]

One goal for neutrino physics is to develop a model that explains neutrino masses, and relates them to masses of other elementary particles. Determining the mass ordering is important as it could be used to exclude many of the newly proposed models[22].

On the other hand, CP symmetry violation is also relevant in neutrino oscillations. The CP symmetry implies that physics is unchanged if particles were replaced by their antiparticles. The breaking of the symmetry between matter and antimatter is called CP violation [23]. Neutrino oscillations are a potential candidate for CP violation as the data for the T2K experiment (a Japanese long-baseline experiment) suggests [23]. CP violation in neutrino oscillations imply that the probability of neutrino oscillations is not the same as the probability of anti-neutrino oscillations [23]. By continuous investigation and observation of CP violation in neutrino oscillations, it can shed light on why the universe has more matter than antimatter.

5 Conclusion

The phenomenon of neutrino oscillations gives us further insight into neutrino physics, for example, the implication of neutrinos having non-zero mass as mentioned in section 3.4 challenges the superluminal speed hypothesis. Neutrino oscillations proposed a solution to the solar neutrino problem and the atmospheric neutrino anomaly detailed in section 3. Questions relating to our universe also arose from neutrino oscillations, such as mass ordering and CP violation as explained in section 4. Overall, neutrino oscillations were a significant scientific achievement and help us understand more behind the ever elusive elementary particle.

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[4]word count: 2892 (overleaf) - 120 (abstract) + $15 \cdot 16$ (numbered big equations) + 66(other) = 3078